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PHYSICAL ASPECTS

of

COLOUR

AN INTRODUCTION TO THE SCIENTIFIC STUDY OF COLOUR STIMULI AND COLOUR SENSATIONS

BY

DR. P. J. BOUMA

PHILIPS RESEARCH LABORATORIES
EINDHOVEN (NETHERLANDS)

WITH 113 ILLUSTRATIONS AND AN APPENDIX

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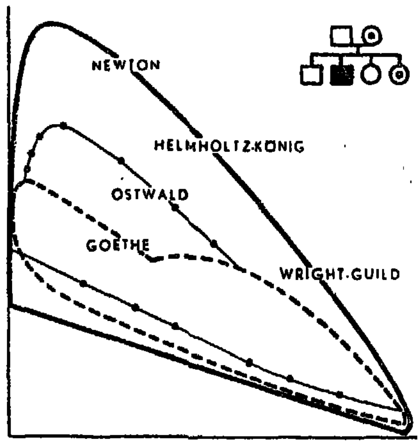

The diagram originally designed by the author for the cover, symbolizes the matter dealt with in this work.
It shows in the colour triangle XYZ of the ICI, the curve of the spectral colours and the purple line derived from the investigations of Wright and Guild. The curve also bears the name of Newton, who, in his colour circle has given the first approximation as well as those of Helmholtz and König, the pioneers of the quantitative elaboration of Young's theory.
The curve of the marginal colours, closely connected with the work of Goethe and the curve of the characteristic colours of Ostwald, are represented as well. The figure in the right-hand top corner symbolizes the theory of heredity which is so important for the explanation of Daltonism.

PREFACE TO THE ENGLISH TRANSLATION

IN September 1944, a few weeks before the liberation of Eindhoven, Dr Bouma, who had been suffering since 1940 from a disease of the central nervous system, which fortunately did not interfere with his keen scientific thinking but gradually bereft him of the control of his body and the powers of speech and writing, communicated to me his plan to collect in book-form his accumulated knowledge in the field of colorimetry and colour science. He started writing with his left hand and finished the manuscript in Dutch in April 1945. Due to various circumstances the printing was delayed until 1946, and in November of that year I was able to place the first copy of his book in his hand. Although he was seriously ill this event gave him great pleasure. On Jan. 19, 1947 Pieter Johannes Bouma died, much mourned by his colleagues of the Philips laboratory and by all those who had met him in his various functions in the field of illuminating engineering.

At the request of Mrs C. J. Bouma-Querner, I set myself the task of supervising the English translation, which was carried out by Mrs A. Wolters-Dawson. I am also much indebted to Mr F. G. Garratt and to Mr C. G. A. Hill for a careful revision with respect to technical expressions. The translation follows faithfully the original Dutch text. A few additions, which in my opinion might increase the value of the book, are marked by an asterisk (sections 33a, 36a, 75, 75a). Sect. 36a has been rewritten after one of Bouma's last papers on the subject (Physica 12, *189, 1946). At the same time I have taken the opportunity of referring in a few places (sections 65 and 67) to recent work of Dr Hessel de Vries (Groningen), which seems to throw a new light on the nature of the fundamental response curves of Helmholtz and their relation to the relative luminosity function. In 1946 I called Bouma's attention to the first of these papers and I am convinced that, had he lived, he would have approved it. I am very thankful to Mr De Vries for his permission and advice.*

It is hoped that Bouma's clear exposition and his simple style, often verging on the naive, will charm the English-speaking reader and that the book in its new form may prove useful to those interested in colour science.

EINDHOVEN, November 1947 — *W. DE GROOT*

PREFACE TO THE DUTCH EDITION

THE subject of "Colour" lies on the boundary of various pure and applied sciences, including physics, physiology, illuminating engineering and chemistry, while it also has several points of contact with art and aesthetics. As a result the same subject may be treated from different points of view. An ideal treatment would take all these viewpoints equally into account. Two practical difficulties prevent the attainment of this ideal: in the first place the treatise would become very extensive and secondly the author does not feel competent to take into account all the different sides of the colour problem. He has therefore restricted himself in the main to approaching the subject from the side of experimental physics and illuminating engineering, the stress in this book falling on the expression of colours by numbers, and on the calculation and measurement of colours.

The physiological and the psychological aspects of the corresponding problems are only considered so far as is necessary for a better understanding of the physical aspects. Here too, where possible, the exposition is based on established experimental facts and the prolific theories are omitted.

The relations with art and aesthetics are not discussed, neither are the physics and chemistry of coloured substances.

This book is intended for all persons interested in the origin and the measurement of colours, such as students of physics, illuminating engineers and technicians, but it is hoped that it will also prove of interest to biologists and physicians (especially Ch. X), and to all those who have to do with the manufacture and use of colorants and coloured products in general.

The reader is assumed to be acquainted with mathematics and physics as far as they are taught in secondary schools and to possess a certain amount of zeal! The parts in small print generally make higher claims but they may be skipped without detriment to the understanding of the main subject.

The purpose of the book is to survey the theory of colorimetry and its applications to practical problems, to enable the reader to carry out the calculations occurring in practice, to facilitate his study of professional

literature and to hand him the tools necessary for a deeper study of other aspects of the colour problem.

The theoretical development follows the path developed by S c h r ö d i n g e r *in 1920-1926, which, in the author's opinion, is the only method leading logically to a complete insight into colour problems. Colour space forms an integral part of this theory.*

Various deep-rooted concepts, the heritage of an historical development which did not always proceed along straight paths, and which are nowadays often an impediment to a clear understanding are either omitted or treated separately in an appropriate place (Ch. XI).

A general survey of the contents of this book and of the chief fields of application are to be found in sect. 6 (pp. 28-29). The reader is also referred to the table of contents.

Finally I have the agreeable task of thanking my wife and a few colleagues, who assisted me by their constructive criticism, and others who by rendering innumerable lesser services made it possible for me to write this book under difficult circumstances. My colleague Dr W. de Groot especially must be named in this connection.

EINDHOVEN, September 1945 — *P. J. BOUMA*

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CHAPTER I

Introduction

§ 1 *Dramatis personae*

Liberation of Eindhoven, September 1944.
Children, gay with orange ribbons, are dancing and singing on a lawn under the shade of a yellow American parachute. Golden-orange sunshine helps us to realize that a new and better era is about to begin.
While musing over this happy scene it occurs to me that all the important and indispensable actors that play their part in the production of *colour* and therefore have their place in this book are now present.
The action of the play is really a succession of events beginning with the golden-orange sunshine and speeding by way of the parachute, orange ribbons, green grass, light and shade, through my eye and its optic nerve to my brain, where it finds its happy ending in my consciousness.
In this first chapter the parts which the various actors take in this play of colours will be sketched in broad outline.

§ 2 *The sun as source of light*

The importance of this actor, whose rôle is often played by understudies, is obvious. Without the sun, or more generally speaking without a *light source*, we should have no light and the eye would consequently be doomed to inactivity. Colours would therefore be entirely out of the question, unless one could count as colours the coloured stars observed on bumping into concrete objects, and similar phenomena. We shall, however, concern ourselves exclusively with colour phenomena caused by *light*. But supplying light is certainly not the only task performed by the light source in the play of colours. This can best be seen by comparing the sun with other light sources.
Sodium light, for instance, already universally known before 1940

through its use in street lighting [see Dorgelo [1]]*), is not capable of conjuring before our eyes any variety of colours worth mentioning; even the most colourful objects all appear yellow and yellowish brown when illuminated by this light.

Mercury light, also employed for various illuminating purposes and known to many people as the light radiated by the artificial sunlight apparatus, gives us a much greater wealth of colour. But it is very striking with this kind of light that various objects are seen in a "wrong" colour; that is to say they cause a different colour sensation from that to which we are used in daylight. Red and reddish objects especially display this colour distortion to a great degree. They usually assume a brownish tint and not infrequently appear almost black.

Light from *incandescent lamps* shows coloured objects in a manner which is much more in conformity with their appearance in daylight. An attentive observer, however, will still see clear differences between these two kinds of light. Thus slight variations of colour — especially in the case of bluish tints — will be more easily observed by daylight than by incandescent light; complexions are clearly different, while draughtsmen are well acquainted with the difficulty of seeing yellow lines on white paper. Finally the difference is also illustrated by the fact that ladies out shopping, before buying material offered for sale in artificial light, invariably desire to see it in daylight.

Sunlight or *daylight* appears in many respects to be the ideal light. It displays a great variety of colours, makes it easy to distinguish slight shades of colour, and the colours of objects round about us obviously look natural.

The fact that *moonlight* and starlight fall so far short in all these respects is due not to the kind of light (this does not deviate much from sunlight; but to our inability to bring coloured objects close enough to the light source. Far too little light ever falls on things from these light sources to enable the eye to distinguish the colours well (see also sect. 4).

What are the physical properties which make some kinds of light specially suitable and others unsuitable for distinguishing colours? This question remained an enigma for centuries until Newton in 1666 got himself a glass prism, "to try therewith the phenomena of colours", thereby paving the way towards the solution of this problem. Soon Newton was able to show unshakably the surprising fact that white sunlight, seemingly so homogeneous, might be conceived as consisting of a series of light rays distinguishable

*) Proper names, suffixed with a number, refer to the list of references; see the appendix to this book.

from one another both by the eye and by various physical properties. We now call these components *spectral radiations* or *primary* colours, the latter name being due to the fact, also discovered by N e w t o n, that these radiations cannot be again divided into components with different properties. To the eye the spectral radiations are distinguished by the *colour sensations* they evoke. They display the well-known series of colours that can be observed in the rainbow: red, orange, yellow, green, blue, violet and all intermediate hues. P h y s i c a l l y they are distinguished by a number of properties and each of these properties is in principle useful to characterize the various spectral radiations.

N e w t o n [1]) chose that property which led him to their discovery, namely the difference in change of direction which the various radiations undergo when they pass from air into glass. He distinguished them, therefore, by their *refrangibility*.

As ideas about the nature of light underwent further development other physical properties came into use for this purpose. When, in the 18th and 19th centuries, the general conviction prevailed that light should be considered as a phenomenon of self-propagating vibrations, and that the various spectral colours were distinguished one from the other by their frequency, it was quite natural to take one of the following three properties to characterize them:

a) *frequency* ν, the number of vibrations per second,
b) *time of vibration* T, the time required for the completion of one vibration,
c) *wavelength* λ, the distance covered by the vibration phenomenon in the time T.

The last quantity, the wavelength, is the one most generally accepted and it will be employed throughout this book. The lengths of the various light waves are very small. They are expressed in the unit mμ – one millionth of a millimeter – or Å = 1/10 mμ. We shall invariably employ the unit mμ. There are various methods of determining the quantities ν, T and λ very accurately, but we cannot discuss them here.

Since the beginning of the 20th century there has been a growing conviction that for an explanation of various phenomena it would be better to consider light as a rain of projectiles each representing a very definite and extremely tiny energy (quantum theory). A new, very mathematical theory (wave mechanics) demonstrated that this was not contrary to the conviction that light in other phenomena seemed to behave entirely according to the old vibration theory. The energy represented by such a quantum is different for each of the spectral radiations, being proportional to ν. Thus in this theory it is natural to use the energy of one quantum as a characteristic property for the various spectral radiations.

TABLE 1

Wavelength	Colour sensation
380-436 mμ	violet
436-495 „	blue
495-566 „	green
566-589 „	yellow
589-627 „	orange
627-780 „	red

There is, of course, a certain connection between the *wavelength* of a spectral colour and the appropriate *colour name.* It must, however, be borne in mind that the wavelength is a purely physical magnitude, which can be objectively measured with great accuracy, while in judging the colour sensation we are concerned with a physiological-optical problem, the result of which will depend on all kinds of properties of the human visual organ. The name given to a colour will depend on the circumstances in which the spectral colour is displayed to the eye, and furthermore two observers in the same circumstances may still obtain slightly deviating results because their eyes may be different and, moreover, because the assignment of a name to a given colour is a question of personal taste. The relationship shown in table 1 must therefore be taken as an approximate division of the colours of the spectrum according to their wavelengths. Thus, for example, the light of 560 mμ will be called 'yellowish-green" by most people. The colour lies in between green and yellow, while, in the opinion of these people, green predominates. The individual diversity of the observers comes to the fore in the fact that there are also people who call this same colour "greenish yellow"; in their case yellow makes the preponderating impression. If the observers are requested to classify the colour under either green or yellow, the first group of people will call the colour green and the second group yellow.

Among Dutch researches on this subject we may mention those of Ornstein[1]), Verbeek[2]) and v. d. Werfhorst[1]). (See also sect. 88 and British Colour Council [1]).

We have gone a little more extensively into this point, as it forms a typical example of the position of most colour problems, being on the border of two sciences: on the one hand physics and on the other hand physiology and, in some cases, psychology. This borderline must always be borne in mind if one wishes to penetrate into the nature of colours. Spectral radiations with wavelengths s m a l l e r than 380 mμ or l a r g e r than 780 mμ are in general invisible to the eye. The former form the ultra-violet rays and the latter the infra-red rays (heat radiation). These limits also display the uncertainty mentioned above.

Apart from individual differences these limits are strongly influenced by the intensity with which the spectral radiations are presented to the eye. The greater the intensity the wider the space between the two limits. Thus investigators have succeeded in making radiations of 313 mμ and 900 mμ visible to the majority of observers [see Graham[1]), Saidman[1]), Bächtiger[1]), Fabry[1]), de Groot[1,2]), Goodeve[1]), Schober[1])]. In practice, however, it is desirable to keep to a definite agreement as to the two limits. For example we might take the wavelengths at which the visibility curve (sect. 12) has fallen to 10^{-5} [Uyterhoeven[1])], namely 376 and 788 mμ, but in future we shall keep to the limits 380 and 780 mμ, these being the limits to which most internationally accepted tables for colour measurement extend.

From what has been said above we can now give a rough explanation of the varying degree in which the various light sources (daylight, incandescent electric light, sodium light, etc.) are capable of producing a diversity of colours.

Physically speaking these kinds of light are distinguished by their *spectral distribution*, or, differently expressed, by the proportions in which they contain the various spectral radiations

Sunlight and incandescent electric light (as well as gas, oil-lamp and candle-light) emit a so-called continuous spectrum, that is to say their radiation contains all visible spectral colours. The other kinds of light mentioned contain relatively more red radiation and less blue radiation than sunlight. Mercury light consists mainly of some spectral radiations with wavelengths 404.7, 407.8, 435.8, 546.1, 577.0 and 579.0 mμ. Sodium light consists practically entirely of light of the wavelengths 589.0 and 589.6 mμ.

In order to see how these differences are expressed in the power to evoke a lesser or greater wealth of colour we must call upon the second actor.

§ 3 *Coloured objects*

In our introductory example these were: green grass, orange ribbons and a yellow parachute.

We have seen that objects in our environment show hardly any difference in colour when illuminated by sodium light, which consists practically of light of one single wavelength.

The orange-yellow light emitted by the light source (see table 1) falls on the objects, is partially reflected and the reflected light falls partly on the eye. When reflected, the wavelength and hence also the colour of the light remains unaltered, whence it may be understood that in the case of sodium illumination no other colours than yellow occur (the occurrence of brownish tints is a complication due

exclusively to special properties of the eye, which will be discussed further in sect. 5)

This consideration gives us at the same time the key to the more general problem of the origin of colours when illuminated by light sources containing more than one spectral colour. We have seen above that the differences in colour of the light reaching the eye directly from various light sources is to be ascribed to differences in spectral distribution. Now when we illuminate two different objects with one and the same kind of light and we notice a difference of colour, it is natural to suppose that this difference must be ascribed to different spectral distributions of the light that reaches the eye via the two objects. This supposition is indeed correct. Each spectral colour radiated by the light source is partially reflected by the object. The wavelength still remains unchanged but the proportion reflected is not only different (a) for the various wavelengths but also (b) for each object. It appears from the first fact (a) that the spectral distribution of the reflected light will generally not be equal to that of the light emitted directly by the light source, and from the second fact (b) follows the confirmation of the supposition that the spectral distribution of the light reflected by the two differently coloured objects will indeed be different.

We can illustrate what has been said by the use of the concrete examples chosen in sect. 1. The yellow parachute reflects the red, orange and yellow rays of the sunlight in a fair degree, the green and blue rays, however, in a very limited degree. The orange ribbons have almost the same properties, except that they do not reflect a part of the yellow quite so well.

Finally, we receive from the green grass light in which the green rays are strongly represented, while the radiations lying at both extremities of the spectrum are considerably weaker.

All light falling on the object that is not reflected is absorbed, that is to say, tranformed into heat.

"*White*" objects constitute a special case. These appear to reflect all rays of sunlight well and to an approximately equal degree. The result is that the light reaching the eye from a white object has approximately the same spectral composition as the light emitted directly by the light source. It is in fact somewhat illogical to label objects with these properties as merely "white" objects, since, if we illuminate them, for example, with sodium light or with the light of a red dark-room lamp, they will make a yellow, or in the latter case, a reddish impression on the eye. The only excuse existing for

this less accurate denomination is the fact that at the time when such an idea was formed people had not yet any great experience of such strongly coloured light sources.

The example of the white objects illustrates most clearly that for the production of colours at least two actors must co-operate: the nature of the light source and the reflecting qualities of the object both play a part. If this fact is borne in mind it will not be difficult to understand a little of the different colour rendering of the various light sources. We have already spoken of sodium light. Mercury light contains no red and orange rays. If a "red object", which is capable of reflecting only red and orange light, is illuminated with this light, it is plain that hardly any light reaches the eye from the red object. In other words, red objects invariably look dark in mercury light, and sometimes almost black. Some of the less conspicuous differences between the colours produced by daylight and incandescent electric light now become clear. The difference in the reflecting properties of white paper and yellow ink, for example, lies chiefly in the fact that the ink reflects the blue and greenish blue rays to a far smaller degree than the white paper. As incandescent electric light, in comparison with davlight, contains fewer blue and green rays the difference in reflecting properties between ink and paper will produce a less obvious colour sensation by incandescent light than by daylight. The ink shows up less against the paper. Yet by no means all colour phenomena can be explained by the co-operation of the two actors mentioned. Thus, for example, it is still inexplicable why "yellow objects", which reflect the red, orange and yellow, as well as a portion of the yellowish green part of the sun's rays, display approximately the same colour as sodium light, which consists only of one spectral colour. Moreover it cannot yet be understood why, in spite of the enormous difference of the continuous spectrum of sunlight and the line spectrum of the mercury lamp, the colour of the light emitted directly by both light sources differs so little that we are inclined to call both "white". This and similar cases point to the fact that yet a third actor plays a very important part in the production of colours. This actor is the visual organ of the observer.

§ 4 *The part played by the eye*

The construction and functioning of the human visual organ is such an interesting and complicated subject that a multitude of books

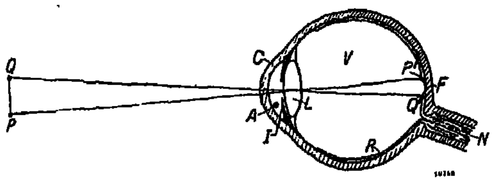


Fig. 1

The human eye (very schematically)
A = aqueous humour; V = vitreous humour; F = fovea centralis; N = optic nerve; C = cornea; L = lens; R = retina; I = iris.

might be written about it, while numerous side issues of the problem of sight are still far from solved.

So when we are compelled to enter the territory of the *physiology of sight*, we must do so with the necessary circumspection We shall only consider here those points which are indispensable to a general survey of the process of the production of colour. In the following chapters we shall, when necessary, give amplifying details. We shall also endeavour to avoid speculations and unproved theories as much as possible, and to build only upon experimentally established facts. The physiologist is chiefly interested in the problem *how* we see, i.e. in the mechanism underlying the various links in the process of sight. The physicist, the technician and in general the practical man, on the other hand, attach most importance to the question *what* we see; that is to say, the final result reached by the chain of processes occurring when the eye is exposed to a given physical situation. Without wishing to go to extremes, we state explicitly that generally speaking it is our desire in this book to consider the problems of sight from the second point of view.

Fig. 1 shows diagramatically a transverse section of the eyeball [for more detailed illustrations see Polyak[1]), Wright[11])]. Let PQ be an object. (For the sake of convenience the distance between PQ and the eye in fig. 1 has been taken far too small, for in reality this distance measures at least 10 times the size of the eye.) We see the object because a pencil of rays starting from PQ strikes the front of the eye (that is the cornea C) and via the aqueous humour (A), the crystalline lens (L) and the vitreous humour (V) reaches the back of the eyeball, the *retina* (R). The various curved planes through which the pencil of rays has passed produce a sharp inverted image PQ of the object, in the same way as the lens of a camera throws a sharp image on the sensitive plate. Focussing at various distances, accomplished in the camera by the shifting of the lens, is brought about in the eye by a change of form of the crystalline lens (L). The quantity of light falling into the eye can be regulated by the opening and closing of the ring-shaped iris diaphragm (I), by which the diameter

of the pupil is made smaller or larger (analogous to the diaphragm of the camera).

Up to this point we have only had a simple optical problem to deal with. But as soon as the pencil of light has reached the retina the whole process is changed; for the various organs situated in or near the retina have the task of transforming the energy of the incident light into another form suitable for carrying the stimulus via the optic nerve to the brain. The first phase of this transformation is of a photochemical nature. The various elements sensitive to light *(rods* and *cones)* contain substances which undergo chemical changes under the influence of light. When the illumination ceases their original condition is restored after a while. These photo-chemical changes are accompanied by the liberation of energy in electrical form. The liberated energy, in a manner still for the most part unknown, causes certain equilibrium disturbances of an electrical nature, which can be transmitted by the optic nerve (N). Finally this stimulus reaches the part of the cortex (regio calcarina) destined for the assimilation of sight impressions, where the most mysterious and enigmatic stage in the production of colour sensations takes place, the commutation of the stimulus introduced by the nerves into a conscious light and colour sensation. Although we are still groping in the dark at many points in the preceding series of processes, it may well be expected that as research proceeds we shall be able to explain all stages of the process as physical and chemical phenomena. But this last stage, the stage of consciousness, lies so far beyond the sphere of science that mankind may never succeed in penetrating its nature.

Scientists have succeeded, chiefly with the eyes of animals, in investigating by measurement two stages in the electric processes occurring in the act of seeing.

In the first place the potential differences between the front and back walls of the eye were measured. This method is especially suitable for investigating transient phenomena. When the eye is suddenly exposed to light the potential is seen to change with a characteristic curve and finally to assume a constant end value. If the light is suddenly switched off, we see the potential return once more to its starting value.

In the second place the electric currents flowing in the separate tissues of the optic nerve were measured when the corresponding part of the retina was stimulated. These currents have the character of a swift succession of equal impulses which follow each other with increasing speed as the intensity of the stimulus increases.

The transition from the former kind of electric equilibrium disturbance to the latter must be imagined as follows:

Between the light-sensitive element and the optic nerve there is among other things an "equivalent network" that works as a condenser. The

observed changes in potential difference depend on the charge of this condenser. When the charge has reached a certain amount the condenser discharges suddenly and this brings about an impulse which propagates itself along the optic nerve.

In surveying the long road from light pencil to colour sensation we must fix our attention chiefly on the retina in order to gain some insight into colour laws.

As we have already seen, the retina contains two kinds of light-sensitive elements, rods and cones, which possess very different properties. In the first place there is the difference in shape, whence the names. Much more important is their different manner of functioning. The rods contain a light-sensitive substance called *visual purple*, the photo-chemical properties of which are already fairly accurately known.

[See for example the investigations of König[8]), Trendelenburg[1, 8]), Hecht[1]), Studnitz[2]), Dartnall[1]), Granit[1]), Ludvigh[1]), Broda[1]), Goodove[1]), Lythgoe etc. etc.].

The sensitivity of the rods is enormous. Rough estimates have shown that in the most sensitive condition of the retina one or two light quanta reacting on a rod chemically are sufficient to produce a sensation of light [de Vries[1])], while ingenious experiments by van der Velden[1]) have made it highly probable that at least two light quanta, absorbed in the visual purple, are required. According to Hecht and others [Stiles[2])] this number is five to seven.

The cones, on the other hand, do not contain visual purple, or only in such a very low concentration that we must look for other chemical substances to explain the light-sensitivity. Only very recently Von Studnitz[1]) succeeded in indicating a light-sensitive substance in the cones of various animals, the general properties of which make it probable that this is the substance we are looking for. The distinction in photo-chemical substances leads to enormous differences in functioning. When the level of illumination is low, as in moonlight and still lower levels, practically only the rods function. With medium and higher levels of illumination, i.e. in artificially lighted apartments and in daylight, it is the cones that function. In between lies a region in which our light sensations are due partly to the cones and partly to the rods, for example on most of our artificially illuminated roads.

A second important difference is that we can observe the various colours with the cones, while the rods do not enable us to distinguish colours. With the rods we see the world actually as a black and white photograph. All objects are white, grey or black. In the dark all cats are grey!

This difference between cones and rods explains the unsuitability of moonlight and starlight for distinguishing colours, hence the gradual fading of colours at dusk.
A third important difference lies in the local distribution of both light-sensitive elements over the retina. When we look straight at a small object, i.e. stare at it, an image is formed in the central part F (see fig. 1) of the retina. About this point extends a practically circular-shaped retinal area of about 0.25 mm in diameter, called the *fovea centralis* or central groove (the retina is slightly depressed here). The fovea contains nothing but cones. It forms the centre of a retinal area called the *yellow spot* or *macula lutea*, of about 2 mm diameter, faintly coloured by a yellow pigment. As we pass out from the fovea through the yellow spot the number of cones per cm^2 gradually decreases and the number of rods increases. This state of affairs also continues outside the yellow spot, so that, far from the centre, in the so-called periphery, the cones are only thinly spread and the rods predominate.
In this local distribution lies the explanation of further peculiarities of sight. Thus in daylight we can observe most accurately with the fovea, because the cones are closest together there. If we wish to examine a detail carefully we stare at it. When the light is feeble conditions are quite different. Small, barely visible objects, for example faint stars, are lost to sight when we try to fix our gaze on them, as their image then falls on the fovea, which contains no rods. If we avert our gaze slightly, say 10°, the image again becomes visible.
It appears from the above that in studying the origin of colours we must confine ourselves principally to cone vision, and may not descend to very low illumination levels, and must therefore pay most attention to events taking place on the fovea, i.e. observations made centrally.

The reader wishing to go deeper into the physiology of sight should refer to Helmholtz[1]), Hering[1]), v. Kries[2]) (more or less antiquated classical works), Schrödinger[7]), Alb. König[1]) (handbook articles), Wright[7]) (short survey), Studnitz[2]) (chemical), Trendelenburg[8]), Polyak[1]), Duke Elder[1]).

§ 5 *The fundamental number three in the colour problem*

We now return to the question raised in 2 and 3, the relation between the spectral composition of light as it strikes the eye and the resulting colour sensation. We have already established the important fact

that a difference in colour between two objects observed simultaneously always goes hand in hand with a difference in spectral composition of the two pencils of rays meeting the eye, but that the reverse is not always true. There are cases where two pencils of light have a radically different spectral composition and yet cause the same colour impression. Of this phenomenon, which, as we shall see, forms one of the foundation stones for the classification of colours, we shall first give a striking example in connection with what was said at the end of sect. 3. The yellow light radiated by some motor head-lamps is caused by placing before the bulb of an incandescent lamp a certain kind of glass (cadmium oxide glass) which allows all rays with a longer wavelength than 500 mμ to pass practically unimpeded, but absorbs almost completely rays with a smaller wavelength. The beam of light radiated, which therefore contains red, orange, yellow and green light, is hardly distinguishable in colour from the spectral radiation with a wavelength of 576 mμ and by a slight variation in the composition of the glass can even assume the colour of sodium light (589.3 mμ).

We may conclude from this and like phenomena that the number of actual colour sensations is very much smaller than the number of spectral distributions we might be able to make.

An altogether different consideration confirms this fundamental fact for us. If we think of all the spectral colours from violet to red plus the transition colours from spectral red via carmine, purple to violet, we have a continuous series of brilliant colours gradually passing from the one into the other. Their essential difference, namely the property of being red, orange, yellow etc., we call their *hue*. But there are of course other colours, the less brilliant, paler, more whitish ones. From each spectral colour, or purple colour, we can pass through colours becoming progressively paler till we approach white. The colour retains its hue throughout, so that, for example, a series of ever paler oranges is obtained. The aspect in which the individual members of such a series differ, namely the property of being more or less white, we call their saturation. The progression from a spectral colour towards white is spoken of as the desaturation of the colour. But the collection of possible colour sensations is not yet exhausted. For, starting from a particular colour, we can modify the colour sensation still further by taking care that the relative spectral distribution (the proportion in which the various spectral radiations occur) remains unchanged, while decreasing the total quantity of light reaching the eye, for instance by removing the light

source farther away from the coloured object under observation. If a similar reduction of light is applied simultaneously to all objects in the field of vision, the character of the colours will change little (provided one does not descend too low; see sect. 4). The *apparent brightness* of the various coloured objects will diminish in equal measure. If, on the other hand, the light is reduced only on one object, while the surroundings remain unchanged, for example by illuminating that one object with a separate adjustable lamp, or by replacing the object by another that reflects one and the same factor of each spectral colour to a lesser degree, a reduction in the apparent brightness will also be observed, but in this case it will often be accompanied by a modification of the character of the colour impression. Thus a red colour will approach black via reddish brown, an orange colour via brown, yellow via yellowish brown, or olive brown, green via olive green and white via grey. If, as is usual in measuring colours, care is taken that only one colour or two slightly varying colours appear in the field of vision, this complication will not occur. In that case the colours brown, grey, etc. will never be met.

We have now varied the colour sensation in our imagination in *three* independent ways, by modifying the hue, the saturation and the apparent brightness. The remarkable fact is that we have now met all imaginable colour sensations. In other words, we can pass from any colour sensation whatsoever to any other one by altering in succession the three magnitudes: hue, saturation and apparent brightness. Expressed differently, a colour sensation is completely determined by these three properties.

The train of thought developed here is not in the nature of a strict argument and will therefore be followed later by a more exact discussion, but enough has been said to convince ourselves how greatly the number of colour sensations lags behind the number of possible spectral distributions. While we can completely exhaust the number of colour sensations by varying only three quantities, it would be necessary to vary the contributions of all spectral colours independently of each other in order to run through the entire number of spectral distributions. The contradistinction "three— all" expresses the relation of the contents of the two families or sets very well indeed. The set of spectral distributions is infinitely richer than that of the colour sensations and one particular colour sensation may, as we shall see later, be evoked by an unlimited number of different spectral distributions.

In the preceding argument the idea of a "set" must not be thought of in the strictly mathematical sense. If all quantities were allowed to vary continuously there would, mathematically, also exist a great distinction, namely the power of a set of functions of one variable to the power of the continuum. If only a finite number of values are allowed, as both physics and physiology demand, both sets become finite, although even then the set of spectral distributions will be very much more extensive.

§ 6 *Experimental confirmation of the part played by the number three*

An exact proof of the part which the fundamental number three plays in the set of colour sensations can only be furnished by careful experimental research. The results of this research will also show how an exact *measurement* of the colours is possible.
Since in this problem the question is always what colours belong to certain *mixtures* of spectral colours, it goes without saying that the research referred to also concerns the mixing of coloured rays, or, as the usual but less exact expression has it, the mixing of colours. This mixing must of course take place in the experiment in the same manner as the mingling of the spectral components takes place in nature, namely by ensuring that the various colours to be mixed strike the same part of the retina simultaneously. As the actions of the separate rays add up together as it were, we speak here of *additive* mixing.
In contrast to the process of additive mixing there is *subtractive* mixing, also known by the more correct but less usual name of *multiplicative* mixing. By this is understood the changes that the colour of light undergoes when passing two or more coloured filters in succession, or when reflected by two or more coloured surfaces in seccession. *) This process occurs especially in the mixing of colouring materials. Much simpler laws obtain for additive mixing than for subtractive. Henceforth when we speak of "mixing" we invariably mean *additive* mixing.
Furthermore, the experiments alluded to are so arranged that the observer always sees and compares two coloured light spots simultaneously, preferably in the shape of two contiguous semicircles. The two light spots strike the eye with light of quite different spectral compositions. Now the task of the observer is this, that leaving one of the light spots unchanged, for instance the left-hand one, he must alter the spectral composition of the other, right-hand,

*) As *no* mixing of spectral colours occurs in this process it would probably be more correct to speak of subtractive or of multiplicative colour *formation*.

light spot in such a way that the *colour sensation* he receives from both lights becomes identical. Then the two halves of the circle have become indistinguishable, the demarcation line between the two having vanished.

The left-hand colour has therefore been *matched* by varying the other colour in a particular way. In these experiments the eye is therefore employed merely to determine whether the two halves have become indistinguishable, and this form of observation is actually the only one in which the eye can act as an accurate and reliable measuring instrument. The result of such an adjustment to obtain equality of colour sensations is obviously that two otherwise different spectral distributions have been found which produce the same sensation. If we were to collect in a haphazard fashion as many as possible of such pairs, we should obtain an array of facts impossible to evaluate and from which it would be practically impossible to draw simple conclusions. The experiments were therefore more systematically arranged. The system lies in the way in which the spectral composition of the right-hand colour is varied to obtain an identical sensation. Led by a surmise in existence since the time of Newton, the light of this right-hand semicircle is compounded by additive mixing of three previously selected, fixed radiations. Spectral colours can be used for this, for instance a red, green and blue colour, but other coloured lights are also usable.

The only alteration made while adjusting for identity of colour is a modification in the quantities of the three component spectral colours. If this experiment is repeated, adding various other colours in succession in the left-hand half of the field of vision, we obtain the result — a priori by no means self-evident — that with this limited possibility of variation an adjustment that gives identical colour sensations is always possible. (For a few limitations see sections 19 and 21.) This experimental result, which forms one of the basic laws of colour theory, can be expressed thus: *By mixing three selected spectral colours in definite proportions any given colour sensation can be matched.*

This confirms exactly our former statement that "the set of colour sensations can be completely exhausted merely by varying three quantities"

But the result obtained is also of far-reaching significance in another sense, for the quantities of the three spectral colours used for matching can be used to fix and characterize the colour. Two colours of similar appearance, although their spectral composition need not be

the same, can be imitated by the same quantities. Two colours of different appearance, however, will require different quantities.
This *classification* and *measurement* of colours, based on the fundamental law formulated above, will form the pièce de résistance of this book. After an introductory discussion of classification (chapter III) there follows the theoretical basis of colour measurement (IV and V) and colour calculation (VI), illustrated by some interesting examples (VII). Next the practice of colour measurement will be discussed (VIII and IX), while finally in chapter XIV a survey will be given of the aspects of science and technology in which what has been discussed can be put into practice. As a knowledge of the subjects we have just mentioned is absolutely essential for a fairly complete treatment of it, we were obliged to keep this subject for the last chapter. For the sake of some readers who may not feel satisfied with the more theoretical and abstract discussions if they have not gained some idea as to where this all leads and what practical advantage may be gained, we shall anticipate a little on the subject matter of the last chapter. The fields of practical application have been divided into three groups.
In the *first* place (sect. 89 and 90) a thorough knowledge of colours is necessary as a guide to the development of new light sources. This problem has become acute, especially during the last ten to twenty years, since progress in technology has made it possible to construct light sources producing the most divergent colours economically. For example, it is of the greatest importance to the science of *signals* to know whether a certain spectral distribution will produce the intended colour signal for the observer. If the light source is used to *illuminate* our surroundings we are especially interested in the colour sensations the light produces in the eye after reflection by coloured objects. According to the way in which the light source is employed, in dwellings, offices, workshops, on highways, etc. the questions must be studied whether these colour sensations produce a pleasant, natural and tranquil effect, whether the colours enable us to perform certain operations well, whether the kind of light employed makes it possible for us to distinguish fine shades of colour, etc. In all these cases it is of primary importance for us to be able to calculate the colours and to define them in unambiguous numbers.
In the *second* place (sect. 91) the measurement of colours is practised in trade, industry and science. There are often great difficulties attached to the use of colour samples to define the colours required or proffered for certain articles, particularly in the case of perishable

articles. In this case it is of great importance to possess a system in which each colour can be unambiguously defined by a limited set of numbers — as may be surmised, this number is generally three. In industry the measurement of colours is also of importance in another respect. The colour of a product often forms an important indication of its quality. Finally, in scientific work the colour produced by a particular phenomenon can be indicated much more exactly by certain numbers than by an inevitably vague description of the nature of the colour sensation.

In the *third* place the theory and practice of colour measurement are practically indispensable for the development of technical processes for the purpose of *reproducing* colours: colour prints, colour photography, colour films, etc. "Trying out" and the critical assessment of the results do indeed play a great part here, but without the guidance of colour theory it would be a mere groping in the dark, which would never lead to the best results. Besides the pièce de resistance just mentioned, in the remaining chapters we shall deal with a number of other subjects which are of interest for the production and judgment of colour impressions.

§ 7 *Comparison of the set of colours with other three-dimensional sets*

In sections 5 and 6 we have been able to show, both by speculation in general and by accurate experiments, the special rôle the number three plays in the survey of all colour sensations. To summarize the result once more: "The set of colours is three-dimensional" It is of value to give a few other examples of similar collections which can be entirely exhausted by varying three quantities and in which each of the members can be designated by particular values of these three quantities.

In the first place we take the number of points in space *(fig. 2a)*.

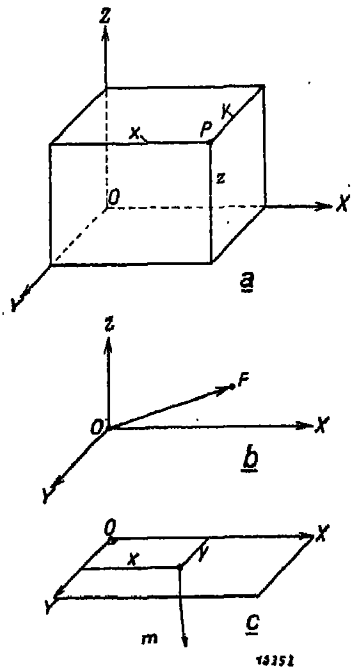


Fig. 2
Three-dimensional sets.
a. Points P in space; b. Forces OP applied at a point O; c. Weights *m* placed on a horizontal plane XOY.

To fix any given point P in space choose three fixed planes OXY, OYZ and OZX which meet at the axes OX, OY, OZ and of which the origin O is the common point of intersection (one can imagine these three planes as being the floor and two adjoining walls of a room). The distances x y z may serve to mark the point P exactly and it is clear that by varying these three quantities independently of each other any point of space can be reached.
In the second place we choose the set of forces which may act at a given point O *(fig. 2b)*. After having added, as above, the three planes with their axes OX, OY and OZ any given force acting at O can be represented by an arrow the direction and length of which agree with the direction and magnitude of the force. The position of the end P of the arrow determines clearly the magnitude and the direction of the force acting at O. As the set of points P is three-dimensional, this is also the case with the set of forces.
In the third place we choose the set of weights that could be placed on a horizontal plane *(fig. 2c)*. Each individual of this set is characterized by its place on the plane and the magnitude m of the weight, or if we add two fixed straight lines OX and OY, by the three quantities x, y and m. By varying these three quantities independently of each other we again discover each individual of the set.
Precisely these three examples were chosen because all three will be used in later chapters (the first in discussing colour space, sect. 24, the second in the addition of colours, sect. 26, and the third for Newton's centre of gravity construction, sect. 69).

§ 8 *The physiological significance of the number three*

Now that we have shown the significance of the number three in colour theory, it is natural to ask the physiologist whether in the morphological examination of the eye properties have come to light which give evidence of a corresponding triple activity of the eye. The answer to this question is disappointing. Various *theories* have indeed been devised to explain the occurrence of this number. According to the oldest theory there were supposed to be three different types of cones reacting to light in different ways. The one type was supposed to be sensitive chiefly to rays from the red end of the spectrum, the second to those of the centre, the third to those of the blue end of the spectrum. The presence of these three types of cone would fully explain the experimental results. The strengths to which these three types of cone were stimulated would form the

three independent magnitudes by which a colour sensation is determined. But in spite of a century of progress in the examination of the retina not a single trace has been found of the existence of these three kinds of cones, nor has any triplicity been discovered in the separate fibres of the optic nerve. Other arguments also bear witness against this original supposition. Thus the colour of very tiny specks of light would be very uncertain and variable, for when the eye makes small involuntary movements the image would fall first on a cone of one type and then on a cone of another type. There was therefore nothing for it but to suppose that each cone possesses separate threefold properties. This supposition has lately received support of an experimental nature, since von Studnitz[3]) claims to have succeeded in discovering in the light-sensitive matter belonging to the cones three components which react to light in different ways. The situation might be compared with cones each possessing three different kinds of photo-electric cells, and two colours would then only be judged as equivalent when the three cells supplied the same current in the two cases. A similar equivalent mechanism (a schematic representation of a system, which need not be quite correct but which reacts externally in exactly the same way as the organ it represents) would indeed entirely explain the experimental data (sect. 53), but the aforementioned investigations into the existence of three kinds of light-sensitive substances in the cones are so uncertain and are contradicted so repeatedly by others [Polyak[1]), Granit[1,2]), Le Gros Clark[1]), Wald[1])], that it would be premature to try to draw definite conclusions.

These speculations form a striking illustration of what was said at the beginning of sect. 4 about "what" and "how". It is clearly apparent from experiments *what* we see, but on the question *how* these impressions are produced we are still for a great part in the dark. For practical purposes, therefore, we shall go farthest if we confine ourselves mainly to the question "what".

§ 9 *Apparent brightness as one of the characteristics of colour*

In the survey given in sect. 5 of the set of colour impressions, one of the three characteristics takes a special place, namely the apparent brightness While variations of the other two quantities modify the character of a colour, this is not the case when the apparent brightness is varied (strictly speaking this statement is only correct when we consider one colour at a time or vary the apparent brightness

of the surroundings at the same time). Colours which only vary in brightness occur frequently round about us. Think of a table top unequally illuminated by a reading lamp, the play of light and shade on a lawn and, finally, the various objects in a scene illuminated by sodium light. In all of these examples the light rays that strike the eye from various parts of the field of view have the same relative spectral distribution (in the second example only approximately) but contain various quantities of light. And in every case it appears that colours which differ only in brightness are called by the same name. The table is brown both in its well-lighted parts and in its shadowy parts, and grass is green in the shade too. Among the three colour characteristics discussed in sect. 6 (i.e. the quantities of the three selected colours required for matching) there is no characteristic that takes such a special place. But we can easily see that brightness plays a simple part also in this system. Let us consider two colours, which need not be similar either in their colour impression or in their relative spectral composition, and which we display in succession on the left-hand side of the field of vision. If we suppose, further, that the three characteristic numbers for the one colour are proportional to those for the other colour, this means that we can match them by producing on the right-hand side of the field of vision successive mixtures of the three selected spectral colours which have the same mixture proportions. But in that case the two right-hand colours have the same relative spectral composition and therefore differ only in brightness. The same also holds, therefore, for the two original colours (left-hand side of the field of vision). So we find that two colours differ only in brightness when the three characteristic numbers for the one colour are proportional to those for the other colour.

However the system of arranging the set of colours is chosen, the brightness is invariably found to occur as a special characteristic. On account of the great practical and theoretical importance of this quantity, both for *colorimetry* and for *photometry* in general, the next chapter will be devoted solely to the conception of brightness.

CHAPTER II

Brightness

§ 10 *A general view of the subject. The three main concepts*

The subject to be treated in this chapter is largely neglected in books about colour, and in those cases where it is more widely discussed the confusion of ideas is often so great that it is better to skip that discussion in order to understand the rest of the book.

What are the apparent difficulties of this subject? They can be divided into two groups:

1. An attempt is made to define the idea of brightness by means of extensive experimental research; but the trouble is that the results of the experiments vary exceedingly according to the manner in which they are arranged. Consequently it is impossible to decide which of the results is the correct one on which to base the concept of brightness.
2. Frequently the word "brightness" is used for a series of different ideas which are indeed partly related to each other but yet differ so much that the ambiguities lead to great confusion.

The difficulties of the first group seemed at first so insurmountable that Schrödinger in 1920 expressly suggested building up the theory of colour without making use of the concept of brightness, and showed at the same time that this was indeed possible. Theoretically this course, which has been adopted before, seems very attractive, but in practice the concept of brightness keeps cropping up, and the connection between colorimetry and photometry is so close that such a separation is not to be recommended. Moreover since 1920 the difficulties have been partly overcome, so that we prefer to take the bull by the horns and to analyse the concept of brightness in such a way that in the following chapters we shall know precisely what is meant. The difficulties of the second group are only to be avoided by introducing different names for different conceptions. If this principle were to be fully carried out we should have to invent five or six names to cover the various senses in which "brightness" has been used. For our purpose, however, three will be sufficient.

a. *Apparent brightness.* This is the more or less vague concept employed in the previous chapter. It corresponds directly to our original, intuitive conception of brightness, and cannot be further defined by any more or less arbitrary agreements. It is inseparably connected with the properties of the eye: a blind man has no conception of brightness, and apparent brightness can vary from person to person.

b. *Brightness.* A concept for which "physical brightness", "objective brightness" or "normalised brightness" are perhaps better names. We shall, however, abide by the name "brightness", as this name and its foreign equivalents have been fixed by international agreement. This latter concept is in many respects the exact opposite of the former. It has been strictly and completely defined by a series of more or less arbitrary conventions. It is pre-eminently suited for making calculations. Although it has grown in the course of time from the former concept, it has lost during this evolution so many of its original characteristics that in its present form it is a property of the radiation from a surface and contains practically nothing more of the co-incidental properties of the human eye. Even a blind man can in principle measure brightness, and the brightnesses measured by different people on the same object must be identical, otherwise one at least has measured inaccurately.

c. *Subjective brightness.* In the course of development between (a) and (b) we can distinguish various intermediate stages. They all differ from concept (a) because they are based on various, more or less arbitrary conventions, which make them to a certain extent suitable for calculations. They all differ from concept (b) since, in defining them, the various properties of the eye are taken into consideration to a far greater degree. All these intermediate stages we shall group under the name "subjective brightness".

§ 11 *Apparent brightness*

The human eye is to some extent capable of judging and comparing apparent brightness. How far does this ability go? Let us consider two spots of light, for instance the same two adjacent semicircles, radiating light of the same relative spectral composition. If we observe the two spots of light simultaneously we find that the eye is able to make the following decisions:

1. to decide whether the two spots have the same apparent brightness.

2. to decide which of the two spots, if they are not equal, is the brighter,
3. very roughly to decide whether the difference between the two spots is great or small.

The first of these points is the most important. As we have supposed the relative spectral distribution to be the same, we can, merely by varying the quantity of light from one of the spots, make the two pencils of light — physically speaking — equal. In this case the apparent brightness is also the same. This is the same case as discussed in sect. 6: the appearance of both spots has become indistinguishable and the demarcation line has vanished.

If we compare the case of sect. 6 with the present case we see that the eye plays the same part in both, but that the equality of the two halves of the field of vision is arrived at in different ways.

In the experiments described in sect. 6 this was attained by varying *three* characteristics of one of the two halves independently of each other. This is the typical process occurring in most colour measurements, i.e. in *colorimetry*.

In the experiments dealt with at the moment equality is attained by varying only *one* magnitude, namely the total quantity of light of one of the halves. This is the typical process which occurs in most measurements of light quantities, i.e. in *photometry*. The numbers "three-one" give the impression that photometry is actually a much simpler science than colorimetry; that light measurement is merely a simple special case of colour measurement. But this conclusion is not generally correct, for we have presumed hitherto that the relative spectral composition of the two spots was the same. In photometry it often happens that two lights of quite different spectral composition are to be compared, for instance the light of a sodium lamp and that of an incandescent lamp. In that case the two halves cannot possibly be made indistinguishable by simply varying the quantity of light; a difference in colour always remains. The eye now has the much more difficult task of ascertaining whether two u n e q u a l colours have t h e s a m e apparent brightness or not. This process is called heterochromatic photometry. This difficult task for the eye does not occur in colour measurements in which we only have indistinguishability to decide upon. In this respect therefore photometry has difficulties to overcome which do not exist for colorimetry. These difficulties led S c h r ö d i n g e r to suggest keeping the two sciences strictly separate (sect. 10).

Is the eye really able to compare the apparent brightness of two

different colours? More accurately expressed, supposing we leave the left half unchanged and vary the total quantity of light in the right half, which emits a light of another colour, is the eye then able at a given moment to decide that "*now* the apparent brightness of both halves is the same"? From time immemorial opinions on this question have been divided. Some eminent workers in this field believe that such adjustments have absolutely no meaning, and that in the process of seeing the apparent brightness and the colour are so closely connected that it is impossible to judge the equality of the one while ignoring the differences in the other. And yet this is what is required of the eye in this case. When one asks oneself whether the apparent brightness of a red flower is greater or smaller than that of the surrounding green grass, one may be inclined to support this opinion. However, if one tries without prejudice to carry out the experiment, the following experiences are encountered.

1. The adjustment can be made so that the right half undoubtedly shows a stronger apparent brightness than the left half. An adjustment with the opposite effect can also be easily made.
2. The greater the colour difference the farther apart these two adjustments lie, if the decision greater or smaller is to be made without hesitation.
3. For the unskilled these two adjustments lie far apart (when the colour difference is great there may be a factor of 5 to 10!). The interval between both adjustments can be considerably lessened by practice.
4. Such an interval also occurs in comparing the brightnesses of homochromatic colours. When a satisfactory adjustment to indistinguishability has been made the quantity of light in one half may be slightly increased or decreased (1 or 2 %) before a clear distinction between the two halves can be observed (see also chapter XII). An experienced observer makes the equality adjustment in the middle of this interval.
5. A similar method is also possible in the comparison of heterochromatic colours. The interval is made as small as possible and the adjustment is made in the middle. As the size of the interval increases with the increasing colour difference, the accuracy and reproducibility will be much less in comparing widely differing colours than when they are equal.
6. In comparing unequal colours the result is far more dependent on slight differences in the arrangement of the experiment, in the

method of adjustment, in the psychological state of the observer, skill in abstracting oneself from the colour difference, consciously or unconsciously applied artifices for the attaining of this, etc. These influences cause a considerable deterioration in the agreement between different observers and also between the observations of one and the same person on different days.

To sum up we can say that adjustment to equal apparent brightness is certainly possible with different colours (heterochromatic photometry), although the accuracy and precision of the adjustment obtained in comparing homochromatic colours can by no means be attained.

These considerations are of course only valid when the intensities are high enough. When the intensities are lower all the difficulties described disappear along with the colour differences (sect. 4).

We shall see later in the further study of the properties of colours that in certain cases the eye has a task to fulfil which is entirely analogous to the adjustments of heterochromatic photometry (sect. 82).

One further word about the case in which the two spots of light do not appear to the eye simultaneously but *in succession*. If they are shown immediately after one another practically the same considerations apply as before. If, on the other hand, some time elapses between the displaying of the two spots a comparison of the apparent brightness is hardly possible. In the first place our memory for apparent brightness is fairly weak, but besides this there is the possibility that the condition of the retina (sect. 4) has altered in the meantime. Thus we may even in certain circumstances judge the same spot of light differently when it is shown twice running with a fairly large interval, if, for example, the sensitivity of the retina has altered in the meantime.

The conception "subjective brightness" introduced by Wright[8]), which, in spite of its name, is fairly covered by our term "apparent brightness", takes this variable sensitivity of the eye into account. In order to compare the apparent brightness of two colour sensations which reach the eye in quite different circumstances, Wright allows one impression to strike the left eye and the other the right eye (binocular matching). This method, invented by Hering[3]) and refined by Wright[4]), is especially suited to giving us an idea of the changes of sensitivity in the eye, but if we wish to use the method for quantitative measurements the necessary care and precautions must be observed. Wright always did this, but some investigators have applied the method in a manner which probably leads to incorrect results [Frieser[2])], while others have employed it in problems of glare [Schouten[2])] which is certainly not justified.

We shall in future restrict ourselves to the case in which the two light impressions to be compared occur simultaneously on adjoining parts of the retina, so that we are not concerned with unequal sensitivity of the retina.

§ 12 *Relative luminosity curves*

The purpose behind our development of the concept of brightness in this chapter is to be able to calculate unambiguously the resultant brightness from physical data on the light source and the reflecting surfaces etc. If we were not able to succeed in this purpose it would, for instance, be impossible to determine the quantity of light emitted by a given incandescent lamp. A step which brings us a little nearer to this purpose, and which is still based solely on the concept of apparent brightness in the comparison of two spots, is the determination of the relative luminosity curves [G i b s o n [5]), J o n e s [4])]. We know already that the eye is not equally sensitive to all spectral colours. Its sensitivity is high for colours in the middle of the spectrum, but towards the extremities of the spectrum (red and violet) the sensitivity gradually diminishes and eventually reaches the vanishing point (infra-red and ultra-violet).
Expressed differently, in order to attain the same apparent brightness we shall have to transmit much more *energy* per second of violet light into the eye than of yellow light. As this wording of the matter contains the expression "the same apparent brightness" it is obvious that the human eye will be capable (see the beginning of sect. 11) of investigating the problem of eye-sensitivity experimentally and give us the result in numbers. This is indeed the case.
In a purely physical sense the quantity of a certain radiation which reaches the eye is expressed in *energy*, which can be measured by transforming all the light into heat and by determining the increase in temperature imparted to a certain body as a result of that quantity of heat. But we are only interested in the quantity of light which reaches the eye *per second*. If we wish to express the radiation in physical units we must measure the energy radiated per second. This quantity is called the *power* radiated. As a unit of power we can use the *watt*, the same unit as is used in the theory of electricity.
The problem can now be expressed in the following form: in the left half of the field of vision a colour sensation is produced by causing a power E_0 of a selected wavelength λ_0 to fall on the eye. Now we throw light of another wavelength λ on the right half and ascertain

what power E of this new light species must be used to make the apparent brightness of both halves equal.

The eye has therefore the same task to *fulfil* here as that described in sect. 11: an adjustment to equal apparent brightness of two differently coloured lights. Once the adjustment has been made, the power E_1 must still be measured, but this is a purely physical measurement about which we cannot enter into details here. After the experiment described above has been repeated several times in succession for different values of λ, a series of powers will have been found which must be used in each case in order to obtain the same apparent brightness. If, for instance, for one wavelength the power required is three times as great as for a second, we shall agree to call the sensitivity of the eye for the first wavelength one third of that for the second wavelength. The sensitivity is therefore inversely proportional to the power required in the experiment.

The results of a series of such experiments can therefore be reproduced by a curve showing the sensitivity to various wavelengths. Since in practice we are usually interested in the *relative* sensitivity to the various wavelengths, it has become the general practice to multiply all values of that curve by the same constant, so that the maximum of the curve receives the value 1. *Fig. 3a* shows such a curve measured at very *low* intensities, therefore in the area where only the rods are active. *Fig. 3b* is the result of measurements at *high* intensities, i.e. where only cones play a part. It is true that both curves show a maximum value 1, and the sensitivity is seen to diminish at both ends of the spectrum. We draw special attention, however, to the *differences* between the two curves. In fig. 3a the maximum lies at 513 mμ (blue green) while in fig. 3b it lies at 555 mμ

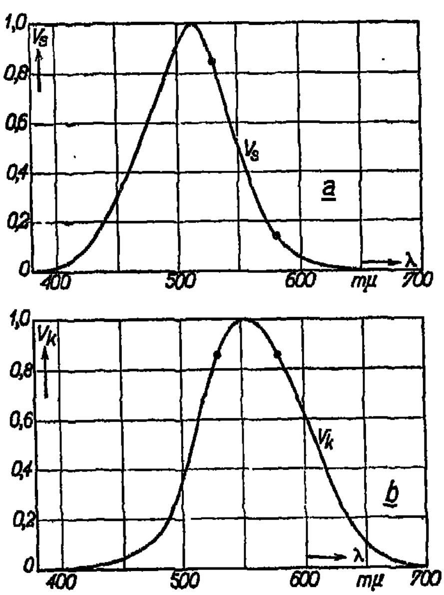


Fig. 3
Relative luminosity curves. a) At very low brightness levels (rod vision), $V_s(\lambda)$ after Weaver[1]; b) At high brightness levels (cone vision), $V_k(\lambda)$.

(yellowish green). The curves as a whole have been shifted with respect to one another. The difference on the red side is particularly great: the cone-curve V_k extends much farther into red than the rod-curve or scotopic curve V_s.

Referring to sect. 4 we therefore find a fourth distinction between the two light-sensitive elements in the eye: they have different spectral sensitivities. If the sensitivity is measured with intensities at which both cones and rods come into action, curves are found which lie between those of fig. 3a and 3b.

Finally we must observe that V_s is better and more unambiguously fixed than V_k, as in determining V_s we are not impeded by colour differences. The uncertainties in V_k are, however, insignificant as compared with the difference between V_k and V_s.

§ 13 *The Purkinje effect*

It follows from the considerable difference between the curves V_s and V_k that the relative brightness in different coloured objects alters when the total quantity of light is greatly decreased. We shall illustrate this with an example. Let one half of the field of vision be strongly illuminated with yellow light of 581 mμ and the other half with green light of 530 mμ. It is clear from fig. 3b that the eye sensitivity for these two wavelengths is equal. If we make the apparent brightness equal we have automatically made the powers equal also. If we descend to lower illumination levels, for instance by lowering *both* powers by a factor 1000, the powers of the two halves have indeed remained equal but the eye sensitivity, which must now be read from fig. 3a, has become quite different for the two lights, namely 0.134 and 0.84 respectively. This means that after lowering the illumination level we should have selected the power of the yellow light about 6 times higher than that of the green light if we wished to attain equal apparent brightnesses. Consequently, with *equal* powers the green light has a much stronger apparent brightness than the yellow light. (Here we employ the second and third determinations of the eye summed up in sect. 11.)

If we had performed the same experiment with red and green spectral colours the effect would have been much stronger.

A similar phenomenon also occurs in nature when we compare different coloured objects first by strong light and then by very feeble light. The effect is indeed less strongly pronounced here than with spectral colours (the objects always reflect other rays as well,

which generally weaken the phenomenon) but it is still very pronounced for whoever pays heed to it. The most striking example is again the combination red-blue, but also with red-green (red tulips in the grass) the effect is still very striking: the red object becomes almost black, as the rods are so insensitive to red light (fig. 3a). The effect just discussed was first described in writing by Purkinje [Purkyně[1]] the Czech physiologist (although it will have struck many people, of course, before him) and is therefore generally called the Purkinje effect. Even the cause — the shifting of the luminosity curves — which at the time of Purkinje was still undiscovered, is referred to as the *Purkinje effect* by many authors.

The Purkinje effect plays a great part in the construction of the concept of brightness and of colour theory, and also in street illumination [Bouma[4, 5], Luckiesh[1]], in the illumination of the dark room [v. Liempt[1]], and in astronomy, etc. The three latter subjects cannot be discussed here.

§ 14 *The summation law*

By determining the relative luminosity curve under various conditions (sect. 12) we have solved the question: "When do two colour impressions produced by different lights possess the same apparent brightness?"

As in practice we are usually concerned with colours consisting of several spectral colours we must also put the same question for these composite light species.

We can of course carry out the experiments described in sect. 12 for all possible composite lights, but this is a hopeless task considering the enormous number of possiblo combinations. A far better way therefore is to try to find a law by which the results of such experiments can be foreseen from the spectral composition of the light and the sensitivity of the eye for the composite spectral colours. Such a law can then be verified experimentally by a number of divergent cases.

In sect. 12 we attained equality of apparent brightness by projecting a spot of light of wavelength λ_0 in the left half of the field of vision which struck the eye with a power E_0, while the right half emitted light of wavelength λ_1 with a power E_1. The eye sensitivity was defined as being inversely proportional to the powers required in this experiment, i.e.

$$V_0 : V_1 = E_1 : E_0 \qquad \text{or} \qquad V_0 E_0 = V_1 . E_1 \tag{1}$$

Equation (1) forms the condition with which the powers must comply in order to arrive at equal apparent brightnesses right and left. This condition may be extended in an obvious manner to composite lights. Leave the left half of the field of vision unchanged but replace the right half by a combination of two spectral colours with wavelengths λ_2 and λ_3 which produce respectively the powers E_2 and E_3 and together effect the same apparent brightness as the left half. Let us suppose that the two lights λ_2 and λ_3 have been, as it were, added up by the eye, then in this case the condition $V_0 E_0 = V_1 E_1$ must evidently be replaced by

$$V_0 E_0 = V_2 E_2 + V_3 E_3 \tag{2}$$

If the light of the right half contains further components λ_2, λ_3, λ_4 λ_5 with the powers E_2, E_3, E_4, E_5, then the condition for equality of apparent brightness becomes

$$V_0 E_0 = V_2 E_2 + V_3 E_3 + V_4 E_4 + V_5 E_5 + \ldots$$

or more concisely

$$V_0 E_0 = \sum_n V_n E_n \tag{3}$$ *)

If the light contains a continuous spectrum we may, without committing any noticeable error, think of it as replaced by a great number of spectral colours (for instance lying at distances of 10 mμ from each other in the spectrum) each with its own power. This shows that equation (3) can be applied in this case as well.

> Actually the expression $\int V(\lambda)\, E(\lambda) d\lambda$ should be substituted for $\Sigma\, V_n E_n$ when passing from a line spectrum to a continuous spectrum. In practical calculations, however, we replace the integral again by a summation. We sum the wavelengths which lie, say, 10 mμ apart

In equation (3) we have found the required law for composite light which enables us to foretell from the various contributions of the components, assisted by the relative luminosity curves, whether the apparent brightness will be the same as that of a fixed spectral colour with wavelength λ_0 and power E_0.

Naturally those values must be taken for the various V's which were found by comparing the power E_0 of wavelength λ_0 with the various other spectral colours: we must therefore always use the

*) Σ (pronounce sigma) is the Greek capital S, which is generally used to denote a summation.

relative luminosity curve appropriate for the level of brightness of the particular light sensation.
This summation law, however, is based on an hypothesis regarding the manner in which the retina adds up the various components of the light, and must be experimentally verified. On this point an extensive investigation has been made by the author on those brightness levels in which the rods assist entirely or partially [B o u m a [1])]. In this domain the validity of the summation law could be accurately established.

Another way of formulating the summation law is the following: when a ray with power E_1 (spectral distribution 1) and a ray with power E_2 (spectral distribution 2) effect the same apparent brightness this also holds for the mixture $\alpha E_1 + (1 - \alpha) E_2$ in which α is any number between 0 and 1. This method of formulation was used in testing the summation law. V a n K r e v e l d [1]) formulated the corresponding law for the photographic plate analogously.

In the region of pure cone sight deviations from the summation law may occur in certain circumstances

§ 15 *Subjective brightness*

We have advanced quite a long way towards our ultimate object merely by the help of our conception of "apparent brightness", as formulated at the beginning of sect. 12, but by itself it can take us no farther. For the illuminating engineer will not only ask the question "under what circumstances are the brightnesses equal?" but also "what is the relation of these two brightnesses? What is the relation of the quantities of light radiated by these two lamps?" And to these questions the eye can give no answer (sect. 11) because it is by its nature incapable of determining the relations of apparent brightnesses. We can now only proceed by agreeing to certain conventions. Here we enter the region of *subjective brightnesses*. Seeing that these conventions are more or less arbitrary it is not to be wondered at that in the course of time several concepts have been introduced which all bear the character of a subjective brightness, but yet show individual important differences.
In order to achieve a logical convention we shall examine more closely the questions of the illuminating engineer. In asking about the relation of the quantity of light from two lamps he generally means: "by how many lamps of the one kind can I replace one lamp of the other?" And when he hears that the brightness of a table surface is a factor two too low he will expect that by adding a second lamp all will be well.

It is clear from this that theory can best be joined to practice by defining brightness for each particular kind of light as proportional to the *power radiated.*

If it were merely a question of one kind of light there would not be the least objection to this, and since the eye is incapable of estimating with any accuracy the relative brightness of two spots of which the one radiates twice as great a power as the other, it is natural *in definition* to call the subjective brightness of the one spot also twice as great as that of the other spot. But difficulties appear as soon as we consider two *different* lights, and those difficulties are for the greater part directly connected with the Purkinje effect. In order to realize this we must return to the experiment of sect. 13. One half of the field of vision radiates yellow light of 581 mμ and the other half green light of 530 mμ, the power as well as the apparent brightness having been made equal for right and left. We will now lower the powers of both halves by a factor 1000. The two powers therefore remain equal. But when we define the subjective brightnesses of *both* colours proportionally to the radiated powers, both subjective brightnesses have decreased by a factor 1000 and are therefore still equal to each other. The apparent brightnesses have, however, as we have seen in sect. 13, become considerably different. We now find ourselves, therefore, in the unpleasant situation that the general convention that "subjective brightness is proportional to power" no longer corresponds in general with the property so much desired that "the apparent brightness is equal when the subjective brightness is equal". The only way out of this dilemma is to give up one of these apparently conflicting desires. For half a century there has been dissention as to which of these demands shall be retained, and although this dissention has officially been settled, the symptoms still appear now and again in the literature.

For physiologists and all who investigate the properties of the eye the second desire is the most precious one. "Of what use is a conception of brightness", they say, "that assigns equal values in circumstances in which the eye observes an obvious difference in brightness?" But the first desire is dearest to the heart of the engineers. "Of what use is a conception of brightness to us", they say, "which does not accept the double value when I hang up twice as many lamps?"

Something may be said for both arguments, but most perhaps for the first. This argument was employed by König, who in 1891 defined subjective brightness as follows. "For the spectral

colour 535 mμ the subjective brightness ("Helligkeit") is proportional to the power. In order to determine the subjective brightness of another spectral colour compare it with the spectral colour of 535 mμ and call the subjective brightnesses equal if the two apparent brightnesses are equal."

This definition is complete and complies with the second requirement, and with the first in so far as that is possible without falling into contradictions.

On the grounds of this definition König[7]) was the first to measure relative luminosity curves with very divergent subjective brightnesses. (The wavelength 535 mμ therefore plays the part here of λ_0 in sections 12 and 14.)

In later years this definition was saved from oblivion, amplified and applied notably in the field of street lighting [Bouma[1, 3, 4, 5])].

The choice of the comparison colour (535 mμ) is of course arbitrary. We cannot discuss further other definitions with other comparison colours, built up in the same way [Dresler[4]), Bouma[3]), Weaver[1])] or, while retaining the "second requirement" supplemented in another manner [Voet-Mogendorff[1]), Bouma[3])]. A survey is to be found in Bouma[9]).

§ 16 *Brightness*

Although, as appears in the previous sections, it has been possible to form a definition for subjective brightness which is at once scientifically correct and practical, engineers have in the meantime not been idle in their efforts to get their wishes realized.

This controversy has not always been conducted along logical lines, but we shall endeavour to arrange the arguments as systematically as possible.

A good example of this controversy is given in the discussion carried on in 1934-35 between Bertling[1]), Reeb[1]), Dresler[1]), Bouma[3])

The difficulties with the concept of subjective brightness originated chiefly in the Purkinje effect, which is in turn due to the shifting of the relative luminosity curve (sect. 12) occurring in the transition from cone to rod vision. If we wish to obtain an attractive definition of brightness from a practical point of view, we must first eliminate the chief cause of complications, i.e. the *rods*. In other words we must confine ourselves for the present to the domain of cone vision. This is undoubtedly not the only domain that interests

the illuminating engineer but it is certainly the most important one. It includes daylight illumination and artificial illumination indoors. The problems of colour vision are also chiefly concerned with this domain. As soon as rod vision begins to take part in the process noticeably, the colours begin to fade and we shall see afterwards that the simple laws governing the theory of colour measurement no longer obtain. If we confine ourselves to the domain of pure cone vision and measure the relative luminosity curve under various levels of brightness by the same methods, we shall always find approximately the same curve having the character of that of fig. 3b.

But if we have exorcised the Lucifer of the Purkinje effect we have on the other hand opened the door to the Beelzebub of colour differences. In the domain of pure cone vision the various colours are so pronounced that the difficulties discussed in sect. 11 (particularly those mentioned under 6) arise in the highest degree. In order to arrive at the satisfactory agreement between the various investigators required in order to obtain a standardized, generally recognized concept of brightness, it was found compulsory to seek another method of brightness comparison. The following requirements are asked of such a method:

1. It must not deviate in its results too much from the results of direct comparison (as in practice hitherto).
2. It must be accurate and reproducible.
3. It must produce relative luminosity curves which are independent of brightness level in the domain of cone vision.
4. It must conform to the summation law.

After some fruitless attempts two methods were found which under certain conditions [Ives [2]] not only fulfilled the requirements but moreover agreed fairly well in their results. These were the *flicker method* [Ives [2]), Nutting [2]), Coblentz [1]), Reeves [1]), So [1])] and the *step by step method*, [Hyde [1]), Gibson [1]), Ives [2])]. By means of the flicker method the two spots of light to be compared are imaged in turn in quick succession on the same part of the retina. Now as the eye has the property of being slower in the reception of colour impressions than of light impressions, the possibility exists of setting the speed of the change so that the eye only observes the difference in brightness but not of colour. In these circumstances an adjustment to equal brightness can be made while eliminating the troublesome colour difference.

By means of the step by step method the colour difference is avoided

or reduced to a minimum by comparing two greatly differing colours not with each other directly but by introducing intermediate colours forming a series from one colour to the other, the difference in colour between two successive steps being very small and if possible so small that the eye does not observe it.

These two satisfactory methods having been discovered it was possible to standardize an average of the best measurements obtained by these methods with a great number of observers, as an *international relative luminosity.*

This decision was taken in 1924 by the international organization for illuminating engineering known in various countries under the following names:

France, England, Holland etc.: C.I.E. = Commission Internationale de l'Eclairage.

America, etc.: I.C.I. = International Commission of Illumination.

Germany etc.: I.B.K. = Internationale Beleuchtungs-Kommission.

The sittings of the C.I.E. have taken place in the last 20 years in Geneva (1924), Saranac Inn (U.S.A. 1928), Cambridge (1931), Berlin-Karlsruhe (1935) and Scheveningen (1939).

The international relative luminosity curve usually referred to as $V(\lambda)$ or V_λ is reproduced in fig. 3b, while the numerical values are to be found in table D under column $\overline{Y}_\lambda = V_\lambda$. On the curve $V(\lambda)$, the conception of brightness and all other illumination and photometric units are based.

§ 17 *The definitions of brightness and other photometric quantities*

"The brightness of a certain light is proportional to the power radiated per cm^2 of the surface (observed in the direction of the observer). For comparing different lights the international luminosity curve V_λ is employed."

Mathematically this definition of brightness B is as follows:

$$B = K \Sigma V_\lambda E_\lambda. \tag{4}$$

in which the summation must be made over all wavelengths of the field of vision occurring in the light, while E_λ and V_λ represent respectively the power and the international relative luminosity curve.

If a continuous spectrum is concerned the wavelength range is divided into sections, say 10 mμ in size, and E_λ then represents

the power radiated in the section between $\lambda - 5$ and $\lambda + 5$. The constant K occurring in (4) (maximum luminosity factor) is determined by the convention that a certain value is ascribed to the brightness of a well defined standard light source. If E_λ is expressed in watts per cm^2 the value of K is then about 660. The unit in which B is expressed is called the *stilb*.

For the standard light source the following have been chosen in the course of time: various kinds of candles, oil lamps (in Germany right up to 1941!), electric lamps (since 1907). Since 1941 we have an internationally accepted standard, namely a black body at the melting temperature of platinum the brightness of which is fixed by definition at 60 stilb (see also sect. 43) The still existing uncertainty about the value of the constants c_1 and c_2 in the Planck formula entails a fairly large uncertainty in the value K in (4). If c_2 is varied from 1.432 to 1.439, K varies from 636 to 688 [Heller[1]), de Groot[3]), Dresler[6])]. According to Birge[1]) c_1 = 3.7403, c_2 = 1.43848, from which Caldin[1]) calculates K = 683 [see also Harding[1])]. Most tables for the calculation of colours are based on the value c_2 = 1.435.

The introduction of this brightness definition has, in principle, completely modified the aspect of photometry. Where there were formerly endless discussions as to the question which *method* of photometry was the "correct" one, in other words corresponded best with the properties of the "average eye", the answer to this question is now "a photometric method is correct if it produces results in agreement with formula (4)."

In the new definition all individual differences between the eyes of the observers have been eliminated. Of course the eye can still be used as a measuring instrument for determining brightnesses provided one is convinced beforehand that the method complies with the requirement just mentioned, within the limits of accuracy which the particular measurement demands. Recently scientists have been trying increasingly to eliminate the eye as much as possible from photometric science [Ives[4]), H. König[1]), Voogd[1])]: there is a growing tendency to substitute *physical* photometry for *visual* photometry. In future when we refer to the conception of brightness we shall do so in the sense of equation (4).

We cannot discuss here the many investigations on the relative value of the various visual methods of determining the relative luminosity curve as well as the validity of the summation law. The following are a few examples from the abundant literature on the subject. Apart from the articles quoted in this section see also H. König[2, 3]), Weigel[1]), A. Kohlrausch[1]), Arndt[1]), Fedorow[1]), Rieck[1]), Dresler[1a, 3]), Preston[1]), Forsythe[1]), Jainski[1]), Jaggi[1]), Pieron[1]), Crittenden[1]), etc. etc. Gibson[5]) and Dresler[5]) give good surveys.

Various other concepts from photometry and illuminating engineering are connected with the concept of brightness, as can be seen from the following.

We obtain the *luminous intensity* from an object (light source or illuminated surface) in a certain direction by multiplying the brightness in that direction by the area seen from this direction. The luminous intensity is therefore a measure for the *quantity of light radiated per second* in a given direction. The unit of luminous intensity is still designated by the old-fashioned title of *"candle"* and is the intensity of 1 cm^2 of surface that has the brightness of 1 stilb. Instead of saying: the surface has a brightness of 1 stilb, we can therefore also say, the surface radiates 1 candlepower per cm^2. Therefore: 1 stilb = 1 candlepower/cm^2.

The *illumination* of a surface is the quantity of light which it receives per m^2 in 1 second. The unit of illumination is the *"lux"*, that is the illumination that a small surface would receive from a light source at 1 meter distance with a luminous intensity in the direction of the surface of 1 candlepower. The *luminous flux* is the total quantity of light radiated by a light source or received by a surface per second. The unit is the *"lumen"*, that is the light flux that a surface receives per m^2 when under an illumination of 1 lux; 1 lux = 1 lumen/m^2.

Now take a light source with a luminous intensity of 1 candlepower in all directions. This illuminates an imaginary sphere of 1 meter radius surrounding the light source as centre with an illumination of 1 lux. But the surface of the sphere is 4π m^2, so the sphere receives altogether a luminous flux of 4π lumen. For a light source radiating an equal quantity of light in all directions we find the luminous flux (in lumens) by multiplying the intensity (in candles) with the factor 4π. If the light source does not radiate equally in all directions then the total flux is 4π times the *average luminous intensity*.

Besides the units mentioned the following are also often used: decalumen = 10 lumen; candlepower/m^2 = $\frac{1}{10{,}000}$ candlepower/cm^2 = $\frac{1}{10{,}000}$ stilb; the millilux = $\frac{1}{1000}$ lux.

There are a number of other units in use, especially for brightness, but their use is inadvisable. [See vocabulary Ned. Sticht. Verlichtingskunde [1])].

We see, therefore, that we can deduce the most important photometric concepts and units directly from the brightness, and that all these concepts are therefore also based on the international relative luminosity curve.

§ 18 *The standardized conception of "brightness" and the lowe brightness levels*

To conclude this chapter we would like just to return to the domain where the rods also (perhaps even exclusively) play a part. May we in this domain, where the conception of brightness defined in sect. 17 comes into conflict with our natural conception of "apparent brightness", also work with "brightness" or must we return to "subjective brightness"? There are various points of view for this question. Our view is as follows: if we wish to ascertain the properties possessed by the eye at lower brightnesses by measurement, we should begin by defining unambiguously the physical circumstances (power, spectral distribution of the light sources, etc.). For this purpose the concept "subjective brightness" with its dependence on all kinds of accidental circumstances and on the choice of the observer is not suitable. If it is desired to fix the starting point by a brightness as well, only the conception of "brightness" as defined in sect. 17 can hold. The experimental investigation consists usually of examining the reactions of the eye while varying one or more physically defined circumstances in a measurable manner. As an illustration see *fig. 4* [B o u m a [5]], giving the results of the work of K ö n i g (see end of sect. 15).

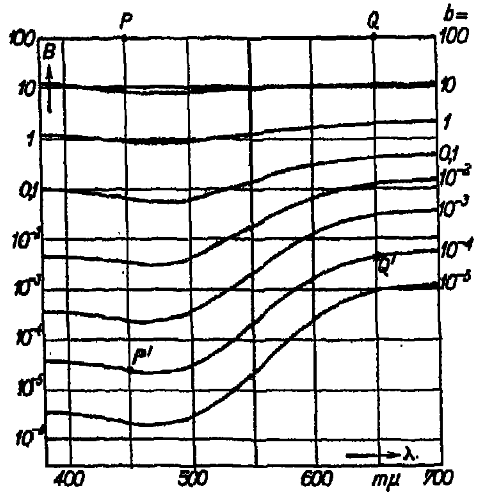


Fig. 4
Subjective brightness (after König) for different values of λ and of the brightness *B*. The difference between PP' and QQ' illustrates the Purkinje phenomenon.

Along the axes are given: brightness *B* and wavelength of the spectral colours. Each point of the diagram therefore represents a physically defined condition (defined by a given *B* and λ) and at each of these points the appropriate measurements of K ö n i g (namely the value of the subjective brightness *b* introduced by him) can be plotted.

If the points of equal *b* are then joined by curves we obtain fig. 4. From this the P u r k i n j e e f f e c t can be plainly seen. If, for

example, we take the extreme case of having to compare blue light of 450 mμ with red light of 650 mμ we see that in order to obtain a subjective brightness $b = 100$ of both kinds of light we must take the same brightness (points P and Q), while to obtain a subjective brightness $b = 10^{-4}$ of the red light (Q') a brightness approximately 200 times greater must be taken than that of the blue light (P'). The points P and Q lie in the area of pure cone vision, while for P' and Q' the light sensation is caused almost exclusively by the rods. If for all research purposes brightness B is chosen as starting point the results of the various investigators can be better compared than if they are based on the conception of subjective brightness, which varies from circumstance to circumstance.

CHAPTER III

The Colour Triangle

§ 19 *The possibility of the representation of colour sensations in a plane diagram*

Our aim is to obtain a comprehensive survey of the apparently chaotic collection of all colour sensations. In order to attain this end it is usually profitable to represent the set by a diagram; in other words to formulate laws and relationships whereby each member of the set has its own place at a given point of the diagram. Definite relations between the various members may then usually be illustrated by simple geometrical theorems.

Is such a representation of the whole colour field in a planar diagram possible? If we recall the considerations of sections 5-7 we see at once that the answer to this question must be in the negative. For in order to encounter all colour sensations we had to vary three magnitudes independently of each other: the set was three-dimensional. Now the manifold of all points in space is also three-dimensional (fig. 2a), that of all points in a plane surface two-dimensional (fig. 2c), that of all points on a straight line one-dimensional (it is only necessary to vary the distance from a selected origin to reach all points).

It appears immediately from this that the only possibility of combining *all* colour impressions in a geometrical representation is to represent them as points in *space*. This representation we shall actually deal with in chapter IV. But both human powers of imagination and of drawing (the page of a book being merely two-dimensional) seem to be better suited to representation on a plane surface. We shall therefore deal with such a representation in this chapter.

As a plane diagram contains too few points to give each colour sensation a separate place we shall have to represent various colour sensations by the same point. We shall therefore make the obvious decision to represent by one point all those colours which only differ in brightness, and which therefore can be made indistinguishable by varying the total radiated power.

It is said of such a set of colours that they belong to the same *colour species* or that they have the same chromaticity. So while in space

we can represent all colours separately, the colour plane is a representation of all colour species or chromaticities. The decision taken has this advantage above other equally possible conventions that all colours with different characters (see sect. 9) are represented in the diagram by different points. The convention chosen, which has reduced the three-dimensional collection to a two-dimensional one, does not therefore prevent us from obtaining a complete survey of all chromaticities.

There are, however, various other decisions to be taken if we wish to obtain an unambiguous representation:

1. Everything must be referred to the *normal* eye, or rather to an average of a great number of eyes which are fairly normal.
2. We presuppose that the colour under observation is present in the field of vision either alone or with a comparison colour differing very little from it (for instance in the shape of two semi-circles, and excluding brown, olive-green and grey tints, as these only occur through the influence of other colours present in the field of vision (sect. 5).
3. We presuppose that the eye is not tired, so that the condition of the retina is not influenced by previous light and colour impressions.
4. We presuppose that the brightnesses observed are all above a certain limit (about 3 candlepower/m^2 = 3.10^{-4} stilb) so that we are assured of using pure cone vision.
5. For accurate measurements, the two semi-circular spots must not be too large, so that we see them in a visual angle of about 2°: in this way practically the fovea alone is in action (sect. 4).

These conventions, which involve a certain standardization of the concept of a *colour sensation*, are to some extent comparable with the conventions which led us from the somewhat uncertain concept of "apparent brightness" to that of "brightness" (chapter II). We shall, therefore, when these conventions are used, no longer speak of a colour sensation but merely of a *colour* or *stimulus* [B o u m a [15])].

§ 20 *The origin of the colour triangle* [see also B o u m a [6])]

We shall again proceed from the experimental result which has already served us in sect. 6 and which we there called one of the fundamental laws of colour theory·

"By mixing three selected spectral colours in definite proportions, we can match any desired colour sensation."

We shall only apply this fundamental law under the conditions just mentioned. The law does indeed hold over a much wider field [G u i l d [5]), B o u m a [1])], but only leads to simple conclusions in the limited field.

We now take for the three "selected spectral colours" those with the wavelengths 700 mμ (red), 546.1 mμ (green) and 435.8 mμ (blue) *) and represent those colours in the colour plane (*fig. 5*) by the points R, G and B respectively. Now in order to define the other colours in the diagram we make a *general rule* that we shall represent the mixtures of two colours by the points on the line connecting the two points representing those colours.

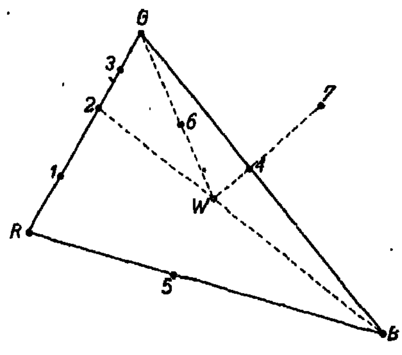


Fig. 5
The colour triangle formed according to our convention (mixture collinear with its components). R, G and B are primaries, W is the white point.

Thus the line RG, for instance, contains all colours that can be matched by a suitable mixture of R and G.
The place of the mixture on the line is determined by the proportion in which we have to mix the two components: for instance the greater the percentage of colour R we have to take the nearer the point is to R.

Quantitatively we can arrange everything so that this convention never leads to discrepancies. We shall return to this quantitative side of the problem in chapters IV and V (see the formula derived from fig. 23).

When we proceed from R along a straight line to G we traverse a continuous series of colour mixtures between red and green. These are orange (point 1), yellow (2) and yellowish green (3). In the same way various blue-green tints lie along the line GB (*e.g.* 4), forming a gradual transition from green to blue, and on the line BR lie the series of purple colours (*e.g.* 5), which brings us back to our starting point R.
Hitherto we have only allotted places in the figure to those colours that could be matched by mixtures of two of the primary colours. These colours (1-5) all lie along the outline of the triangle RGB. Colours which can only be matched by using all three primary colours will be found i n s i d e the triangle.

*) The colours G and B chosen here are two spectral colours occurring in mercury light.

An example will make this clear: seek the place of *white* in fig. 5 (more accurately: the colour that a white object assumes in daylight). To reproduce this R, G and B are all required. We now mix R and G in the required proportion. This mixture is given by point 2. White is now produced by a suitable mixture of the colours 2 and B. According to our convention the white point W therefore now comes to lie on the line 2B. In this way it appears that all colours that can be matched by a mixture of R, G and B are represented by points inside the triangle RGB. And on the other hand these colours entirely fill the triangle: each point can be realized by selecting suitable proportions of R, G and B and then applying the construction just used to find the place of W..

To survey the distribution of the various colours over the triangle we shall draw the line WG. A point (6) on this line will now represent a colour which can be matched by mixing the selected spectral green G with a certain quantity of white W. As we proceed from G to W we shall encounter colours that require an ever greater percentage of white. Starting from the deep green colour G we shall gradually meet less saturated (duller, paler, less brilliant) greens until finally, at W, the extremely pale green becomes white. We recognize in this process a correspondence with what was dealt with in sect. 5: the line GW contains colours all possessing the same hue (namely green), but whose *saturation* gradually decreases.

The analogy is not quite complete. While in sect. 5 we spoke of c o l o u r s e n s a t i o n s, experienced by the eye, we mean here properties of c o l o u r which can be ascertained by measurement.

In the same way the line WR contains red hues of increasing saturation and WB the gradual transition from white to B. Proceeding from W to 4 we find an ever deepening blue-green. In contrast to the preceding examples the maximum saturation has not yet been reached at 4. For among the spectral colours we meet with a still more brilliant blue-green colour (7): this latter must therefore be mixed with a certain quantity of white (made paler!) in order to produce the colour of point 4. According to our convention point 4 must therefore lie between W and 7, so that the blue-green spectral colour 7 lies o n t h e e x t e n s i o n of W4 (fig. 5).

A similar consideration now obtains for the other spectral colours. They are all (except R, G and B) represented by points lying slightly b e y o n d the triangle RGB. For that part of the spectrum with a wavelength greater than 546.1 mμ (the wavelength of G) this effect can, however, be practically ignored.

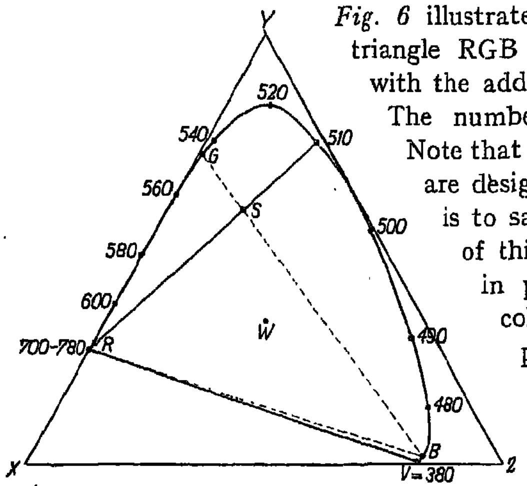


Fig. 6
The complete colour triangle with spectrum locus and purple line RV and the triangle XYZ surrounding both.

Fig. 6 illustrates the situation. The same triangle RGB is again drawn, but now with the addition of the spectral colours. The numbers give the wavelengths. Note that the wavelengths 700-780 mμ are designated by the point R. That is to say the spectral colour species of this region are all represented in point R. All these spectral colours can therefore be completely reproduced solely by a certain quantity of the primary colour R. This fact can be verified by an accurate observation of the spectrum: between 700 mμ and the extremity of the spectrum the colour does not alter. We now draw the line RV connecting the red extremity R of the curve of the spectral colours with the violet extremity (V). The line RV contains, therefore, no spectral colours, but the various purple hues, mixtures of red and violet.

It follows from our general "mixing rule" that *all* colours are represented by points within the area enclosed by the curve of the spectral colours and the "purple line" RV. For all colours are produced by mixing spectral colours, and in mixing we must always remain within the area just described. *)

Now in order to be able to express the various chromaticities by *numbers* we draw an equilateral triangle XYZ completely enclosing the curve of the spectral colours, and fix the place of the point representing a certain colour by the relation of the distances from that point to the three sides of XYZ. In this way each chromaticity is characterized by the *ratio* of three numbers.

This result reminds us of the conclusion of sect. 6, where we found that each colour could be characterized by three numbers, and of sect. 9, where it appeared that for all colours differing in apparent brightness only, and therefore belonging to the same chromaticity, the *ratio* of the three quantities is the same.

*) It is incorrect therefore to call the points outside that curve by the name of colours (unreal, virtual or imaginary colours) as many authors do.

The exact connection between all this will be considered more closely in chapter IV.

§ 21 *The introduction of "negative quantities" of a colour*

Although it is clear from the above how each colour receives its allotted place in the diagram, this explanation still does not enable us to make quantitative calculations regarding the colours. This is chiefly due to the fact that our "general rule" is still rather vague, for we have still to agree on the connection between the situation of a mixed colour and the mixture proportions of its two components (*i.e.* where 6 lies on WG, see fig. 5).

Nevertheless we can draw various general conclusions with the help of fig. 6. In the first place let us look at the colours which found their place o u t s i d e the triangle RGB, for instance the spectral colour of 510 mμ. As we have seen that the mixture of certain amounts of the colours R, G and B always produces a point w i t h i n the triangle, we come to the conclusion that the spectral colour of 510 mμ cannot be reproduced by mixing R, G and B in suitable proportions. Here we have therefore an exception to our "fundamental law", a limitation in its application to which we already alluded in sect. 6.

A careful examination of fig. 6 shows us that we could not have avoided this difficulty by another choice of the three primary colours. For, whichever three points we may select within (or on) the curve of the spectral colours as primary colours, a part of the spectral colours will always fall o u t s i d e the primary colour triangle. We might indeed have made the outside area a little smaller (by for instance selecting for the green primary colour $\lambda = 520$ mμ), but there will always remain exceptions to the fundamental law.

As it is very unpleasant to have to work with a law admitting of exceptions, we shall take care that these exceptions disappear. This convention and the manner in which we shall carry it out will appear strange to many people at first sight. It must be remembered, however, that we shall be doing exactly the same as in many mathematical problems, in which it is also necessary to eliminate unpleasant exceptions! For instance, having fixed the meaning of "to subtract" we are faced with the exception that a greater number cannot be subtracted from a smaller number and when solving quadratic equations we get stuck at $x^2 + 1 = 0$. The exceptions are eliminated, in the first case by the introduction of negative numbers and in

the second by introducing imaginary numbers. The first example shows a great similarity to our problem, for we are going to introduce *negative* quantities of colours. It appears from fig. 6 that the colour 510 mμ can still be simply related to R, G and B. We can, by mixing the colour of 510 mμ with a certain quantity r of the colour R, reproduce colour S. This is in symbols:

$$(510) + r(\mathrm{R}) \longleftrightarrow (\mathrm{S})$$

in which $\longleftrightarrow$ means: "matches" [see also B o u m a [13])]. But we can also reproduce this same colour (S) by mixing a quantity g of colour G with a quantity b of colour B:

$$(\mathrm{S}) \longleftrightarrow g(\mathrm{G}) + b(\mathrm{B}).$$

It follows from these two "colour equations" that:

$$(510) + r(\mathrm{R}) \longleftrightarrow g(\mathrm{G}) + b(\mathrm{B}) \tag{5}$$

or, verbally, if we mix on the left half of the field of vision the colour of 510 mμ with a quantity r of the colour R, and in the right half a mixture of a quantity g of G and a quantity b of B, the line of demarcation will disappear. The equation (5) would indicate that (510) could be matched by certain quantities of R, G and B if the term with R stood at the other side. Then let us bring the term to the other side by writing for (1):

$$(510) \longleftrightarrow - r(\mathrm{R}) + g(\mathrm{G}) + b(\mathrm{B}) \tag{6}$$

and let us then say that we can reproduce (510) by a negative quantity of R plus two positive quantities of G and B.

In this manner the fundamental law holds universally, while the meaning of a negative quantity of a colour has become clear. It must not be mixed with the other — positive — primary colours, but be added to the other half of the field of vision in order to establish colour equation (5) between the two halves of the field of vision. This is the same as the bookkeeper who does not add his debits with his credits but posts them "on the other side" in order to obtain, as with colour adjustments, a balance sheet that tallies!

§ 22 *Dominant wavelength and saturation*

We learn from *fig. 7* (again with the assistance of our "general rule") that we can also reproduce the colour represented by a point K by mixing the white W with a spectral colour (in our case 520 mμ) in a

certain proportion. This reproduction is possible for all points K lying within the curve of the spectral colours, except those lying in the triangle 700-W-380. In this triangle lie the purples, which become the more saturated as we approach the "purple line" 700-380. We can reproduce such a purple shade K′ by mixing the white W with a colour A, lying on the line 700-380. We can also reproduce white W by mixing K′ and a certain spectral colour (here $\lambda = 540$ mμ). We have now found

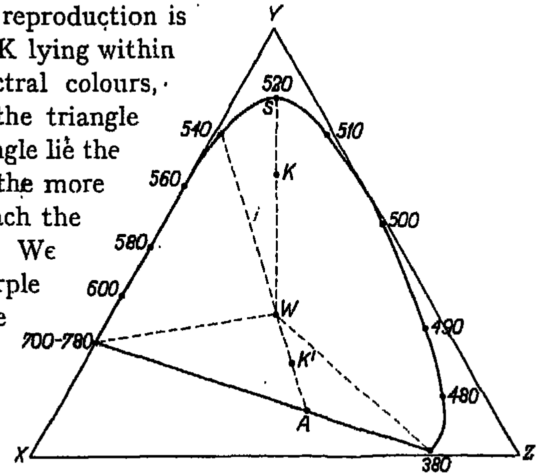


Fig. 7
Dominant wavelength λ_d and colorimetric purity p.
For K: $\lambda_d = 520$, $p = B_S / B_K$.
For K′: $\lambda_d = -540$, $p = B_A / B_{K'}$.

both for colours such as K and for colours K′ an appropriate spectral colour of a certain wavelength, which we shall call the *dominant wavelength*. For the purple shades the dominant wavelength λ_d is the wavelength which, when mixed with the purple shade, can reproduce white, while for the other colours λ_d the wavelength is to be mixed with white to reproduce the original colour.

The expression dominant wavelength is not very happily chosen, for λ_d is by no means always the wavelength predominating in the spectrum (with colours as K′ in fig. 7 this is of course not the case at all). It is even quite possible that the wavelength λ_d does not occur in the spectrum of the colour. The expression has become general however. The term "equivalent wavelength" which is sometimes used for the same idea expresses the sense better — for the non-purple shades at least (compare also the German expression: farbtongleiche Wellenlänge).

In matching a colour by mixing white and a spectral colour not only the dominant wavelength but also a second concept occurs: the degree of saturation or *colorimetric purity, p*.

Supposing colour K (fig. 7) to have a brightness B_K and that a brightness B_S of the spectral colour is necessary for matching by mixing white with spectral colour, then $p = B_S : B_K$ is the purity. In other words p is the relative share that the spectral colour takes in the match.

For the purple colours K′ the purity p means the relative share that

the saturated purple colour A takes in the mixture of W and A. In the language of colour equations:

$$B_K(K) \longleftrightarrow B_S(S) + (B_K - B_S)(W);\ p = B_S : B_K$$
$$B_{K'}(K') \longleftrightarrow B_A(A) + (B_{K'} - B_A)(W);\ p = B_A : B_{K'}$$

The colours for which $p = 1$ obviously lie on the curve of the spectral colours, or on the purple line 700—380. The value of p gradually diminishes to 0 as we approach the centre of the colour triangle, in the direction of W. Given the dominant wavelength of a colour K we can determine on which straight line, starting from W, K stands. Given the purity as well, this means that a selection has been made from all the colours lying on the given straight line, in other words the situation of K on the straight line has been determined. This shows that the place of K can also be fixed by the dominant wavelength and the purity instead of by the relations of the distances to the sides of $\triangle$ XYZ.

The latter system we call the *trichromatic system* and the former the monochromatic-plus white system or *monochromatic system*.

One objection may be raised to the latter system. On the line A-W-540 (fig. 7) for example there are two points corresponding with the same value of λ_d and p; the one lies in the purple area and the other not. A particular example is formed by the colours 540 mμ and A: for both $\lambda_d = 540$, $p = 1$. In order to distinguish between these colours with the same λ_d and p we put a minus sign for the value of λ_d for the colours in the purple area. In this way for 540 mμ: $\lambda = 540$, $p = 1$ and for A: $\lambda = -540$, $p = 1$. This minus sign reminds us, moreover, that a negative quantity of the spectral colour is required for the reproduction of K'.

Now we can determine a colour completely by its place in the colour triangle and its brightness, therefore by its dominant wavelength, its purity and its brightness: λ_d, p and B.

These threefold characteristics of a colour (the number three once more!) remind us strongly of the threefold characteristics we allotted in the general considerations of sect. 5 to colour sensations, *i.e.* hue, saturation and apparent brightness.

For a correct understanding of colour theory it is necessary, however, to distinguish sharply between these two sets of three. We therefore mention the differences once more. The hue, saturation and apparent brightness are intuitive *impressions*; one need know nothing of the colour theory to obtain these impressions. These impressions are very difficult to express in numbers; they are to a high degree dependent

on all kinds of accompanying circumstances, such as the momentary condition of the retina, the presence of other colours in the field of vision, etc. The same pencil of light rays can, in various circumstances, produce quite different sensations. The dominant wavelength, purity and brightness on the other hand are well defined quantities; they can be measured; in order to make the measurements unambiguous the accompanying circumstances can be eliminated by various conventions; the measurement is based on the knowledge of the colour theory (in particular our "fundamental law"). The same pencil of light rays will always give the same λ_d, p and B (provided no errors of measurement are made); we can arrive at standardized values as a result of measurements made (as described in sections 16 and 17), which with the assistance of formulae and tables can be calculated straightaway from the power and the spectral composition of the pencil of rays (see chapters IV - VI). If we recall what was said in sect. 2 regarding the border line character of colour problems we can distinguish between the sets briefly by remarking that the first set of three lies entirely on the psychological side and the second on the other hand almost entirely on the physical side of the boundary.
The same remarks as we made here regarding λ_d p and B of course apply also to the three characteristics already dealt with in sect. 6: the quantities required of three primary colours for the reproduction of a colour form a set of three also lying on the physical side.

§ 23 *The meaning of the word "white"; complementary colours*

The same applies to the word "white" as to the word "brightness": almost everyone uses it without further explanation in the most divergent senses. Although the dangers hidden in such a careless use are not so great here (usually the meaning is quite clear from the context), it might be as well to sum up the various meanings. There are mainly three:

1. A property of the reflecting objects (see sect. 3) which allows them to reflect all spectral colours well and in equal measure. What do we mean here by "well"? There are surfaces, for instance, magnesium-oxyde, which reflect up to 97 % of the light falling on them; good white paper reflects 80-85 %, but in most circumstances we call all objects white which reflect about 60 % or more. If they reflect less they are called grey or drab.
2. The *colour sensation* "white". This, as an intuitive idea, needs no

further explanation. Apart from the spectral composition of the pencil of rays, whether we call a colour sensation white depends on many accompanying circumstances, for instance on the condition of the retina, the influence of surrounding coloured objects, the knowledge that we have an object before us which we should call white in most circumstances (which is for example white in the sense of 1), etc. In other words "calling something white" is influenced by all factors usually influencing a "sensation". In sect. 5 the word "white" is used in this sense.

3. The *colour* (chromaticity) "white". In connection with the exact definition of colours we also require an accurate definition of the colour "white". Thus in sect. 22 we repeatedly made use of the colour "white". In the course of time numerous and varied conventions have been used in this respect. We confine ourselves to those made internationally (by the C.I.E.), which still obtain.
These conventions can be divided into two groups according to the object in view.
In the first place we must define a white colour so that the ideas of dominant wavelength and purity are accurately fixed. For this purpose the "equal energy spectrum" has been chosen. This spectral distribution, designated by E, is one which radiates the same energy per second, that is to say the same power, in every wavelength interval (of for instance 10 mμ) of the visible spectrum.
In the second place there is a need for a standardized light for defining the colour of surfaces. We have already seen (sect. 3) that this depends both on the spectral distribution of the light striking it and on the reflecting properties of the object. Theoretically the only correct method is to define the percentage that the object reflects of each spectral colour. If we note down this percentage for each wavelength we get a curve called the "spectral reflection curve", which establishes the reflecting properties of the object completely. With the aid of this curve we can calculate, as we shall see later, the colour of the object when illuminated by any light of which the spectral distribution is known. But in practice matters are slightly different. The illumination of the coloured object occurs in by far the greater number of cases by means of light sources whose spectral distribution approximates either to that of an incandescent lamp, to sunlight or daylight. Hence it has been decided to define one of each of these groups and always to measure the colour of the

object when illuminated by one of these standard light sources. This has the advantage that now only three numbers need be given for the colour instead of the whole reflection curve. The disadvantage of this method is that we cannot foretell from these three numbers what colour the object will assume when illuminated by a light source differing greatly from the normal (i.e. mercury light). As standard light sources for this purpose (C.I.E. 1931) the following has been agreed:

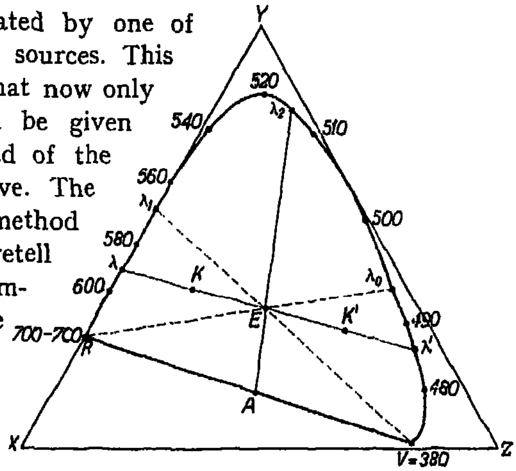


Fig. 8
Complementary colours. K and K', V and λ_1, R and λ_0, A and λ_2 are pairs of complementary colours.

Illuminant A: A gas-filled incandescent lamp with a colour temperature of 2848° K. The colour temperature of an incandescent surface is determined by comparison with the so-called "total radiators", being light sources whose spectral distribution is described by a formula given by P l a n c k. If the incandescent body has the same distribution as a "total radiator" of the temperature T, we call T the colour temperature of the incandescent body. The colour temperature of an incandescent lamp is always a little higher than the true temperature (see also section 42).

Illuminant B (approximating sunlight): the light obtained by transmitting illuminant A through an accurately specified coloured liquid filter in which the colour temperature is brought up to about 4800°.

Illuminant C (approximating to daylight): a similar light source to B but with another filter in which the colour temperature is brought up to about 6500°.

The filter used for B consists of two consecutive layers of liquid, each 1 cm thick, of which the first is composed of copper sulphate (2.452 g), mannite (2.452 g), pyridine (30 cm³) and water (up to 1000 cm³) and the second of cobalt ammonium sulphate (21.71 g), copper sulphate (16.11 g), sulphuric acid (10 cm³), water (up to 1000 cm³). For C a similar filter is used of which the corresponding numbers are however: 3.412 g; 3.412 g; 30 cm³; 1000 cm³ and 30.58 g; 22.52 g; 10 cm³, 1000 cm³. [See D a v i s [2])].

In *fig. 9* and table F are given the relative spectral distributions of illuminants A, B and C.

In contradiction to the standard light sources A, B, C we can imagine E as being replaced by a light source with another spectral distribution but giving the same colour. This is straightaway clear when one recalls the purpose of the standard light sources. To indicate clearly the difference between A, B and C on the one hand and E on the other we sometimes call A, B and C *normal light sources* or *normal whites*, while E is called *standard white.*

There has lately been an endeavour to replace B and C by one light source agreeing in colour with E. No international agreement has, however, yet been reached about the desired spectral distribution. In Germany a third liquid filter has been prescribed for this purpose, to be combined with A [Richter [6]]; Wright [10]) also recommends the same solution.

Directly connected with the choice of the standard white E is the conception "complementary colours". We call two lights K and K′ *complementary* when it is possible to reproduce the colour of standard white E by a suitable mixture of K and K′. With an eye to our "general rule", E lies therefore in the triangle between K and K′ *(fig. 8)*. In particular the two spectral lights given in fig. 8 by λ and λ' are also complementary, as they are in a straight line with E. If we follow the curve of the spectral colours from the violet extremity V to the red extremity R we see the following: the wavelengths between V and λ_0 have complementary colours with wavelengths between λ_1 and R; λ_0 has the red extremity of the spectrum as its complementary colour; the wavelengths between λ_0 and λ_1 have no complementary spectral colour. Thus the wavelength λ_2 has as its most saturated complementary colour purple A; λ_1 has as

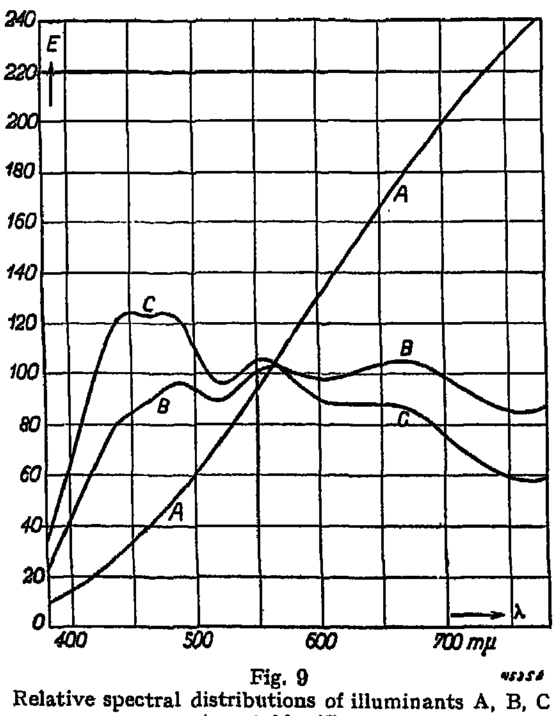


Fig. 9
Relative spectral distributions of illuminants A, B, C (see table F).

its complementary colour the violet extremity of the spectrum; the wavelengths between λ_1 and R have complementary wavelengths lying between V and λ_0.

Table A gives the various pairs of complementary wavelengths (λ and λ'). For the wavelength λ_0 we find 493.9 mμ; for λ_1 we find 569.7 mμ.

It has long been the fashion to search for an empirical formula giving the connection between λ and λ [see for instance Grünberg[1]), Priest[1]), Godlove[1]), Richter[6]), etc.]

Two wavelengths may also be complementary with respect to the colour of one of the illuminants A, B or C [Hardy[3])].

CHAPTER IV

Colour space in its simplest form

§ 24 *Representation of colours in space; Grassmann's first law*

In the previous chapter we have tried with the aid of the colour triangle to survey the position of the colours in respect to each other, which enabled us at the same time to demonstrate some properties of colours and some related concepts.
In many ways, however, these speculations were far from complete. This is because of the "general rule" — made for the sake of simplicity — concerning the situation of the mixtures in the colour plane (sect. 20), and on the other hand it is due to the fact that we were forced time and again to represent a large number of colours by the same point. These two causes are connected. The concurrence of colours with varying brightness made it very difficult to deal *quantitatively* with the mixture laws in a simple way, and this had the further result that we were not able to define the correct position of a particular mixture of two components quantitatively, and it was therefore not at all a matter of course that the "general rule" after repeated applications did not lead to some discrepancy.
Although the colour triangle in the manner dealt with in chapter III has been of great service to us, it is necessary to give now a more general treatment which will do away with the incompleteness of the old method and at the same time be stricter, so that we need no longer fear any discrepancies.
The key to this method lies, as the reader will already guess, in the employment of a geometrical representation whereby each colour has its own allotted point. In sect. 19 it appeared already that this geometrical representation must of necessity be three-dimensional, that is, a representation in space. We must substitute the colour plane by a *colour space*, in which the representation of all c h r o m a t i c i t i e s is replaced by a representation of all c o l o u r s (see also sect. 19).

On the question of whether it is desirable to make use of representations in space in building up the theory of colour, opinions have been divided in the last 50 years. This is due chiefly to the fact that the representation in a colour plane (as in chapter III) is the most suitable for giving the

reader in a simple manner a general survev of the order of colours; if on the other hand it is desired to penetrate deeper into the colour theory and learn to understand and apply the various methods of calculation as well, the use of a colour space is to be recommended.
Helmholtz employed a colour space, but the majority of his followers confined themselves to the colour triangle. Schrödinger in some excellent treatises tried to infuse new life into colour space, but few writers have since ventured to employ spacial representation in their explanation of the basic facts. Various authors did feel compelled [viz. Richter[6]] at a later stage of the construction to have recourse to spacial representation for certain purposes (see sect. 47).
We do not wish to take up the extreme standpoint in this argument that is held by some authors: ".... to attempt to represent the world of colour by a plane surface is patently absurd" [Drever[1]], but we share sincerely the opinion of Ives[3] (1915) that the use of colour space forms "the clearest method of representing the facts of colour mixture"

In sect. 6 and 7 we have already alluded to the way in which we could allot to each colour its place in space. We shall use the "fundamental law of colour mixture" dealt with in those sections. The content of this law was first explicitly formulated by Grassmann[1]) and we shall call it therefore *Grassmann's first law*. We will summarize this law once more in its general form:
If we select three suitable spectral distributions (three kinds of light) we can reproduce each colour completely by additive mixing of these three basic colours (also called *primary* colours). The desired result can only be attained by one particular proportion of the quantities of those primary colours. *)

The conditions under which this law generally obtains are:

a. In taking measurements the general conditions (sect. 19) as to brightness level, size of the field of vision, condition of the retina, and normality of the eye of the observer must be complied with.
b. In certain circumstances negative quantities of a colour (sect. 21) must be introduced.
c. The three primary colours must be selected so that it is impossible to match one of them by any mixture of the other two. This condition is implied by the word "suitable" in the general formulation. A glance at fig. 5 shows at once that this condition is necessary. If we chose, for example R, 2 and G (2 can be reproduced by mixing R and G!) all mixtures of those three basic colours would lie on a straight line and we should never be able to reach the white point W.

It follows from Grassmann's first law that we can characterize

*) Actually the term "primary colours" is incorrect, and ought to be replaced by "primary chromaticities". We shall, however, stick to the term in general use.

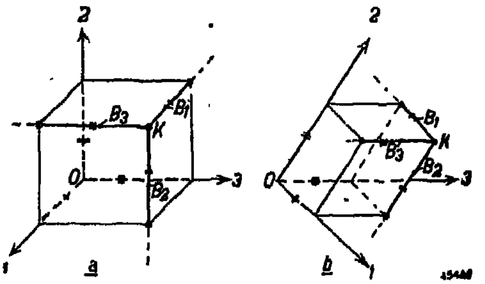


Fig. 10
Characterising a colour by a point K lying at distances B_1, B_2 and B_3 from the coordinate planes. a) Rectangular frame. b) Frame with oblique angles ($\angle 102 = \angle 203 = \angle 301$).

a colour quite unambiguously by the numbers B_1, B_2 and B_3, being the brightnesses (luminances) of the three primary colours required for the reproduction of the colour. In the notation of sect. 21:

$$B \text{ (colour k)} \leftrightarrow B_1 \text{ (col. 1)} + B_2 \text{ (col. 2)} + B_3 \text{ (col. 3)}$$

Let three straight lines O1, O2 and O3 (coordinate axes) be drawn through a point O of space. These three axes can be chosen at right angles to each other *(fig. 10a)* or with oblique angles, for instance by allowing the axes to form three equal angles respectively, smaller than 90° *(fig. 10b)*. Now each two of those axes form a plane that we may call a coördinate plane: thus we have the planes (O, 2, 3), (O, 1, 3) and (O, 1, 2).

If we then choose a point K in space possessing the distances B_1, B_2 and B_3 to those coordinate planes (distance B_2 measured parallel to axis O2 etc.) then we say that point K has the coordinates B_1, B_2 and B_3. Point K therefore represents the colour which is also characterized by the numbers B_1, B_2 and B_3. It is clear that each colour can be given its own point in space in this manner. Otherwise expressed, we have represented the three-dimensional manifold of colours (sect. 6) in the equally three-dimensional manifold of all points of space (see also sect. 7, 1st example and fig. 2a).

In principle it is of no consequence whether we choose the representation of fig. 10a or that of fig. 10b. Henceforth we shall make use of the oblique angled system of 10b, this giving figures which are easier to comprehend.

§ 25 *Position of the spectral colours; Grassmann's second law*

When examining further the position of the various colours in colour space we shall no longer employ more or less arbitrary conventions, but base everything on experimentally established facts. The first experimental result we shall employ is the following:

Grassmann's second law: If two different light spots give the same colour sensation they continue to do so if we increase or decrease

the brightness of both by the same factor (thereby leaving the relative spectral distributions unchanged).

This law too only holds when the general conditions summed up in sect. 19 have been complied with, particularly the conditions of an adequate brightness level.

Consider, for example, any colour K and the mixture of the primary colours which match it. As in sect. 24, we can express the equality of the colour sensations as follows:

$$\mathrm{K} \longleftrightarrow B_1 \text{ (colour 1)} + B_2 \text{ (colour 2)} + B_3 \text{ (colour 3)}.$$

The second law now says that this equality remains when multiplying all the brightnesses occuring in it by the same amount n; in symbols:

$$n\mathrm{K} \longleftrightarrow nB_1 \text{ (colour 1)} + nB_2 \text{ (colour 2)} + nB_3 \text{ (colour 3)}$$

From the two equations we learn that with any change whatever in the brightness of K, the quantities required for the match increase proportionally. But that means that the coordinates of the colour in space also increase proportionally, so that we proceed in space along a straight line through the origin *(fig. 11)*.

Instead of representing all colours differing only in brightness by one point in a plane, as was the case in chapter III, the various members of such a series now take their place on different points of a straight line through the origin. This straight line represents therefore a colour species; its points represent the separate colours. In the neighbourhood of O lie the colours with a low brightness (for the brightnesses B_1, B_2 and B_3 of the matching primary colours are low there too). The further we are removed from O the higher become the brightnesses of the colours (n becomes ever greater).

The relation between the colours K and nK is sometimes expressed by the statement that they represent different "quantities" of the same colour (or colour species), the quantities being in the proportion $n : 1$ (see section 33a, 36a, 69).

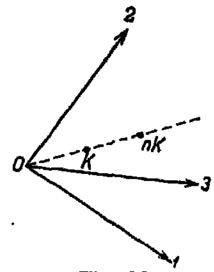


Fig. 11
In colour space a line through O represents a *chromaticity*, whilst the points of that line represent separate colours. O1, O2 and O3 represent the chromaticities which are chosen as primaries.

Fixing our attention particularly on the three straight lines O1, O2 and O3 of fig. 11, we notice that a point in O1 lies in the plane of coordinates O13 as well as in O12. The coordinates B_2 and B_3 (the distances to these planes) are therefore both zero for a point in O1. But that means that

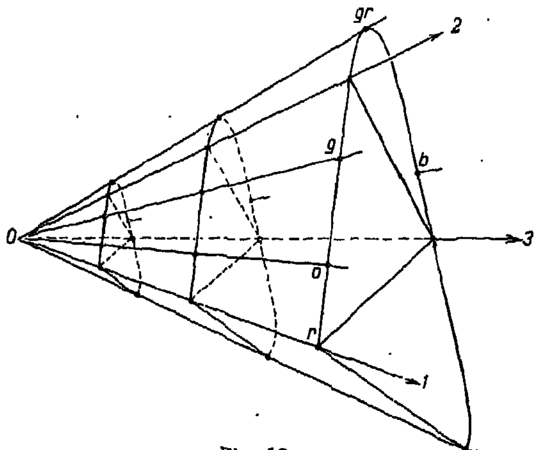


Fig. 12
Colour space constructed with the spectral primaries 700, 541.6 and 435.8 mμ. The spectral chromaticities lie on the surface of a cone containing the axes O1, O2 and O3, which cut the plane on the right along r, o, g, gr, b, v (g = yellow). The plane rOv contains saturated purple.

this colour can be reproduced by using only the first primary colour. Hence it follows that the straight line O1 represents the colour species which we chose as our primary colour. In the same way O2 and O3 represent the colour species of the other two primaries. Each spectral colour is represented by a straight line through O as well.

In order to define the manifold of all these straight lines it is sufficient to determine one point in each straight line. We must therefore fix experimentally the three brightnesses required to match a certain brightness of the spectral colour.

In *fig. 12* these straight lines have been drawn in the diagram. Together they form the shape of a *cone*. To show the shape of this surface sections have been drawn with some parallel planes each cutting off equal parts of the coordinate axes O1, O2 and O3. On one of these sections the positions of the spectral colours red, orange, yellow, green, blue and violet are indicated. Since we have selected, as before (sect. 20), the spectral colours 700; 546.1 and 435.8 mμ for the three primary colours 1, 2 and 3, the axes O1, O2 and O3 also lie on the cone surface.

§ 26 *Mixture of colours in space; Grassmann's third law*

If the coordinates of two different colours K and K′ are known, how are the coordinates of the mixture K + K′ determined? In answering this important question we shall again use an experimental result, namely: *Grassmann's third law.* Two light mixtures which, when displayed next to each other, produce the same colour sensation, act in exactly the same manner when mixed with other lights. They may therefore be substituted one for the other in any mixture.

If K $\leftrightarrow$ B_1 (colour 1) + B_2 (colour 2) + B_3 (colour 3)
and K' $\leftrightarrow$ B_1' (colour 1) + B_2' (colour 2) + B_3' (colour 3)

then we can, according to this law, replace both colours in the mixture K + K' by their matching primary colour mixtures without altering the total colour impression. The formula is:

$$K + K' \leftrightarrow (B_1 + B_1')\ (\text{col. } 1) + (B_2 + B_2')\ (\text{col. } 2) + (B_3 + B_3')\ (\text{col. } 3)$$

We therefore arrive at the simple rule: the coordinates of the mixture are found by adding the corresponding coordinates of both components (*i.e.* B_1 and B_1', etc.). It is easy to see what this means geometrically. In order to find the mixture of the colours represented by K and K' (*fig. 13*) all that need be done is to draw a straight line starting from point K', running parallel to OK and having the same length. The end of this line represents the mixture K + K'. If we recall that each of the coordinates means a distance to one of the coordinate planes (24) we see that after passing down the line OK' we have moved over a distance B_2' from the O13 plane. If after that we pass along the line K' → K + K' we move another distance B_2 further from the same plane. The point K + K' which we have found lies therefore at a distance $B_2 + B_2'$ from plane O13. An analogous reasoning for the distances up to the two other coordinate planes shows that the point K + K' has the coordinates $B_1 + B_1'$, $B_2 + B_2'$ and $B_3 + B_3'$ and therefore indeed represents the mixture of K + K'.

If we remark that in fig. 13 OK(K + K')K' is a parallelogram we can summarize the geometrical "mixing construction" as follows:

From OK and OK' we find the colour point of the mixture by a parallelogram formation exactly as if OK and OK' were two forces to be added to obtain the resulting force O (K + K') (see also sect. 7, second example and fig. 2b).

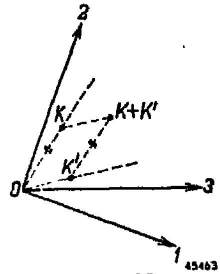


Fig. 13
When mixing two colours K, K' one finds the place of the mixture K + K' by vector addition of OK and OK'.

The contents of Grassmann's laws mentioned here indeed originate from Grassmann, but the formulation in his original publication is quite different from ours. Moreover Grassmann adds two other laws, one on the continuity of colours (which will be taken by many people as self-evident) and another on the additivity of brightnesses which follows directly from the present definition of brightness.

The formulation given here is by no means perfect either. For the sake of simplicity of treatment we have added much more than

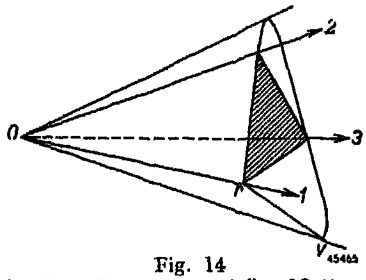


Fig. 14
In the colour space of fig. 12 the colours at the inside of the trihedral angle O,123 have positive coordinates. The other colours have one negative coordinate.

is necessary for the building up of colour metrics. For a rigorous treatment of this question see Bouma[12]).
Grassmann's laws have been extensively tested experimentally by Helmholtz and his followers [König[4]), Brodhun].
If we leave the domain of pure cone vision all three laws show important discrepancies. A theoretical attempt has been made [Bouma[1,2])] to give simple laws in these circumstances as well, but the theory has been insufficiently tested.

It appears in particular from fig. 13 that K, K′ and K + K′ lie in one plane with O. Hence the necessity of condition (c) of sect. 24 again appears. If one of the primary colours could be reproduced from the other two, the three primary colours would lie in one plane with O. But then any mixture of the three primary colours would also produce a colour lying in the same plane and we would therefore only be able to match a two-dimensional collection of colours by such a mixture!
Further it follows from fig. 13 that K + K′ always falls within the angle KOK′. If we apply this rule to fig. 12 we see that in mixing two or more spectral colours we always remain within the cone which is bounded by the spectral colours and the plane rOv. Within this cone each point represents a definite colour; the points outside the cone do not. *)
The plane rOv contains the saturated purple colours, namely the mixtures of the colours forming the red end of the spectrum (Or), and the colours occurring at the violet end of the spectrum (Ov).
In *fig. 14* we see the cone once more with the three coordinate axes O1, O2 and O3 (which again represent the spectral colour species 700; 546.1 and 435.8 mμ).
It will be seen that a number of the points falling inside the cone fall also within the solid angle O,123 (among others, those of the shaded part of the plane of transverse section) but the rest fall outside (among others most spectral colours). The first group of colours has therefore purely positive coordinates; the second group of colours contains one coordinate with a negative value. Thus, for example, for the saturated purple colours B_2 is negative; they lie below plane 1O3; the distance to that plane must therefore

*) It is therefore incorrect to call these points "unreal", "virtual" or "imaginary" colours (sect. 20).

be considered as negative. Here we find requirement (b) of sect. 24 confirmed.

§ 27 *The connection between colour space and the colour plane*

The attentive reader will have noticed many points of resemblance and a great many analogies with what was considered in chapter III. Although the use of colour space is almost indispensable for a satisfactory explanation, yet in practice we almost exclusively meet the much more easily handled two-dimensional representation — the colour plane or colour triangle. We shall now show that there is a very simple connection between these two methods of representation. In doing so we shall be able to apply our well-established knowledge of colour space in order to give a firmer basis to the theory of the colour triangle as well.

In space we defined a colour by the quantities B_1, B_2, B_3. In sect. 25 it appeared that the ratio of these quantities remained unchanged when we merely changed the brightness of the colour. Now if we return to the colour plane, in other words, if we once more represent all colours which only vary in brightness by one point, it is clear that a certain relation of B_1, B_2 and B_3 pertains to each point of the colour plane.

We can now determine this point by the quantities:

$$b_1 = B_1 : (B_1 + B_2 + B_3);\ b_2 = B_2 : (B_1 + B_2 + B_3);$$
$$b_3 = B_3 : (B_1 + B_2 + B_3) \qquad (7)$$

The ratio of the quantities b_1, b_2 and b_3 is apparently the same as that of B_1, B_2 and B_3 and their sum equals 1. Arising from this latter property, it is sufficient to give only two of the three b values (the third can then be directly calculated), so that we have indeed obtained a two-dimensional manifold, which can be represented in a plane, while all colours only differing in brightness (therefore having the same relation $B_1 : B_2 : B_3$) meet in one point.

It is always important to make a sharp distinction between colour coordinates in colour space — often called for short *trichromatic coordinates* — and the quantities defining the colour in the colour plane (often called *trichromatic coefficients*). Therefore we shall henceforth always indicate the first kind of coordinates by means of capital letters (B_1 B_2 B_3 X Y Z R G B etc.) and the second kind by means of small letters (b_1 b_2 b_3 x y z r g b etc.).

There are various ways in which the representation in a plane can be effected.

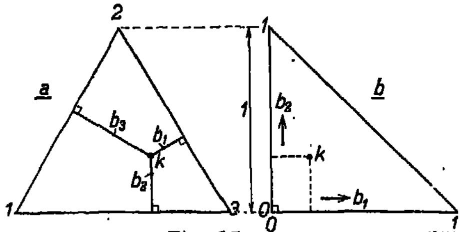


Fig. 15
Plotting chromaticities in a plane using the quantities b_1, b_2, b_3 ($b_1 : b_2 : b_3 = B_1 : B_2 : B_3$ and $b_1 + b_2 + b_3 = 1$). a) In an equilateral triangle. b) In a rectangular triangle.

In the first place *(fig. 15a)* we can draw an equilateral triangle with height 1 and choose the point k for the chromaticity having the distances b_1, b_2 and b_3 from the sides of the triangle. Of course it will have to be proved that this is always possible, so that for each point k the sum of the distances to the three sides equals 1. This, however, is easily seen by noticing that the area of $\triangle$ 123 is the sum of the areas of $\triangle$ k23 $\triangle$ k13 and $\triangle$ k12. The equation is:

$$\tfrac{1}{2}\,(\overline{13}) = \tfrac{1}{2}\,b_1\,(\overline{23}) + \tfrac{1}{2}\,b_2\,(\overline{13}) + \tfrac{1}{2}\,b_3\,(\overline{12})$$

As the sides $\overline{13}$, $\overline{23}$ and $\overline{12}$ are of equal length it follows by division that: $1 = b_1 + b_2 + b_3$.

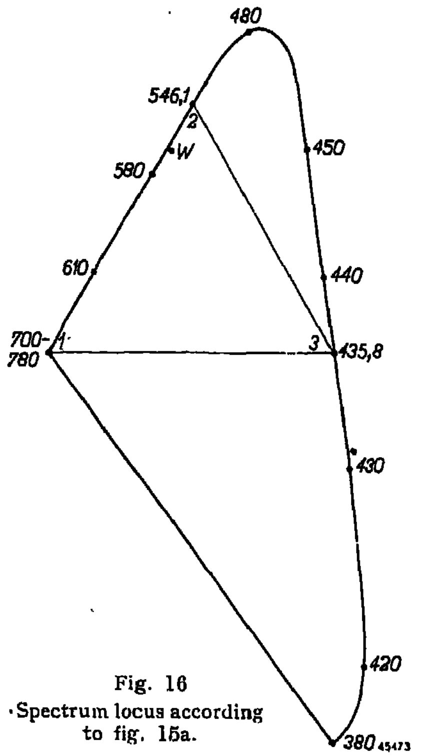


Fig. 16
Spectrum locus according to fig. 15a.

In the second place *(fig. 15b)* we can take a rectangular set of axes and place along the axes two of the b quantities (for instance b_1 and b_2); b_3 can then be calculated directly. Now the quantities B_1 B_2 B_3 are known by measurements for the spectral colours, so that we can also calculate directly the quantities b_1 b_2 b_3 (see table B). If we plot these quantities in the manner shown in fig. 15a we get *fig. 16*. As was to be expected, most spectral colours again fall outside the triangle. In sect. 26 we saw that the majority of these colours had one negative B coordinate, and according to formula (1) this also leads to one negative b coordinate, therefore to a point o u t s i d e the triangle.

The great similarity of fig. 16 with the section of fig. 14 now becomes

obvious, and this similarity is no coincidence. For we can prove that these two figures are completely correspondent. For this purpose we once more draw in *fig. 17* the three coordinate axes and the plane 123. We would recall the fact that the individual angles between the three axes have been made equal and that the plane 123 cuts off equal parts of the axes, from which it follows directly that triangle 123 is equilateral.

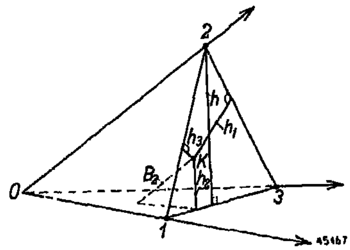


Fig. 17
Relation between colour spaco O, 1.2.3 and the colour triangle 1 2 3.

Further let K be the representation of a colour lying in the plane 123. It can be seen from the figure that: $h_2 : B_2 = h : O2$. If we represent the last ratio by q it appears from the symmetry of the pyramid O.123 that not only $h_2 : B_2 = q$ but in the same manner: $h_1 : B_1 = q$ and $h_3 : B_3 = q$. Hence it follows that: $h_1 : h_2 : h_3 = B_1 : B_2 : B_3$. The distances from K to the sides of the equilateral triangle 123 are therefore proportional to the coordinates B_1, B_2, B_3 of K in the colour space, in other words: the place of the point K in the triangle 123 is fixed in the same manner as the point K in 15a. Hence it follows that in the section 123 of fig. 17 the same representation of the colours does indeed occur as in fig. 16 (the only difference lies in the fact that the figures may still be drawn on different scales; this is, however, of no importance).

We can also express this important result as follows: If O is a small light source, and a few colours K are represented by their points in colour space, then the shadows of those points thrown by the light source at O on the plane 123 will be the representations of the selected colours in the colour plane.

In mathematical language: the colour triangle can be produced by projecting the points of colour space from O on to the plane 123. In fig. 12 this projection is portrayed for the spectral colours.

§ 28 *Conclusions from the connection between colour space and the colour plane*

The connection just discovered enables us to deduce the properties of the colour triangle from those of colour space. In the first place we are again interested in how the mixture of two colours K and K′ takes place in the colour plane. Now we have already seen in sect. 26 that in colour space the mixed colour K + K′ was coplanar with

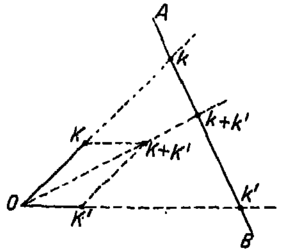


Fig. 18
How the "general rule" of Ch. III follows from the mixing construction of fig. 13.

K, K' and O (fig. 13). This plane has been chosen in *fig. 18* as the plane of diagram. Further AB represents the line of intersection of this plane with the colour plane (plane 123 in fig. 17). The representations of the three colours in the colour plane are k, k' and k + k' — the shadowgraph of K, K' and K + K'.

We see, therefore, that in the colour plane the mixture k + k' lies on the connecting line of k. and k'. This important result, which we formerly (sect. 20) accepted without any scruples, is now proved beyond doubt by the simple properties of colour space.

Plane 123 from fig. 17 possesses one more remarkable property: it is in fact a *plane of constant brightness*. A point K of that plane has the spacial coordinates B_1 B_2 and B_3 and the triangular coordinates b_1 b_2 and b_3, while according to sect. 24: $B_1 = qb_1$, $B_2 = qb_2$ and $B_3 = qb_3$. From this it follows that the brightness $B = B_1 + B_2 + B_3 = q\ (b_1 + b_2 + b_3)$. Now q is the same for each point of plane 123, and the same obtains for $b_1 + b_2 + b_3$. Therefore the brightness B is constant too.

There is, however, one small flaw in this argument. It is undoubtedly true that the matching mixture of the three primary colours has a brightness $B = B_1 + B_2 + B_3$ (this follows from the definition of brightness sect. 17).

But, does it follow that the colour reproduced also has brightness B? In reproducing them we have made the two light spots indistinguishable. This implies that the a p p a r e n t b r i g h t n e s s e s are equal but not that the b r i g h t n e s s e s are equal. For the international relative luminosity curve V_λ was derived from the difficult heterochromatic luminosity measurements, and here we have a quite different and equally subjective measurement. For this measurement it is therefore not certain a priori that equality of apparent brightness implies also equality of brightness. If this should not appear to be the case for the average observer this would form a serious defect in the established curve V_λ and it would be advisable to revise that curve and thereby to take into account the heterochromatic luminosity measurements as well as the colorimetric adjustments. In practice it appears, however, that the defect mentioned does indeed exist to a certain degree but that it leads to only the smallest deviations in all colorimetric adjustments occurring in practice, so that we may consider our argument as correct. It is clearly apparent from this example how cautious one must be as soon as one departs from the precept given by S c h r ö d i n g e r: "keep all brightness measurements quite separate from colour measurements". In this connection one may perhaps wonder if it is not possible to establish a relative luminosity curve exclusively based on colorimetric adjustments, thereby avoiding all

heterochromatic photometry. This appears, however, not to be possible. We can indeed in this manner restrict the use of heterochromatic adjustments to a minimum, namely to the determining of the relative luminosity for the three selected primary colours. See also Guild[4,9]), H. König[4]) and Richter[8]).

We can now calculate where the mixture K + K′ has to lie on the connecting line kk′. It can be proved quite easily by algebraic or geometrical means that (see fig. 18) : (k → k + k′) : (k + k′ → k′) = B' : B *). This expressed in words means that the distances in the colour plane of the mixture k + k′ to the two components k and k′ are in the ratio of the brightnesses B' and B of the two components.

Algebraically this property is proved as follows: let the space coordinates of the two components and their mixture be respectively $(B_1\ B_2\ B_3)$ $(B_1'\ B_2'\ B_3')$ and $(B_1''\ B_2''\ B_3'')$, their coordinates in the colour plane be $(b_1\ b_2\ b_3)$, $(b_1'\ b_2'\ b_3')$ and $(b_1''\ b_2''\ b_3'')$ and lastly their brightnesses be B, B', and B'', then: (k → k + k′) : (k + k′ → k″) = $(b''_2 - b_2) : (b_2' - b_2'')$ (the three colours should be marked in the manner of fig. 15a in the colour triangle). If we fill in here in succession the equalities $b_2 = B_2 : (B_1 + B_2 + B_3) = B_2 : B$; $b_2' = B_2' : B'$; $b_2'' = B_2'' : B''$; $B_2'' = B_2 + B_2'$ and $B'' = B + B'$, the relation sought for will be found by a simple calculation.

Geometrically we derive the property by the aid of *fig. 19* in which the letters have the same meaning as in fig. 17. Draw KK_0 parallel to AB. As AB is the section of the diagram with a plane of constant luminosity, k and k′ (considered as points of colour space) represent colours of equal brightness. The same obtains for the points K and K_0, so that $OK_0 : OK'$ not only represents the ratio of the brightnesses of K_0 and K′ but also the ratio of the brightnesses of K and K′; $OK_0 : OK' = B : B'$.

But now: (k → k + k′) : (k + k′ → k′) = KC : CK_0 =
(K → K + K′) : OK_0 = OK′ : OK_0 = B' : B.

In the special case that the two brightnesses are equal the mixture lies exactly between the colours k and k′. As the "general rule" of sect. 20 now appears to hold in our new colour triangle worked out from the colour space, the conclusions drawn from this in chapter III are also valid without alteration.

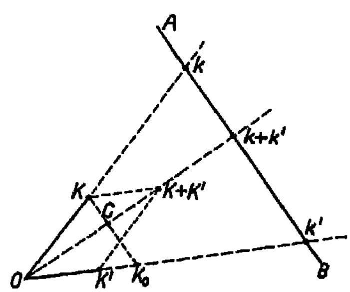


Fig. 19
Derivation of formula (k→k+k′): (k + k′→k′) = B' : B.

§ 29 *Other colour spaces and colour triangles*

The colour triangle 123 found in fig. 16 compared with the triangle XYZ of fig. 6 has two great disadvantages.

*) The sign → should not be confounded with ←→: k → k + k′ simply means the distance between the points k and k + k′.

In the first place several colours have negative coordinates. This property that the triangle has inherited from the colour space of fig. 14 appears in practice to be a great disadvantage: calculating by means of these coordinates is greatly impeded by it.
A second disadvantage appears when we determine the situation of the white point in fig. 16. When we know the position of the spectral colours this determination is made most easily by selecting two pairs of complementary spectral colours (sect. 23). The white point must then lie on both the connecting lines of the two pairs and is therefore found as the point of intersection of these two lines. Now it turns out that the point W lies as shown in fig. 16, close to the yellow spectral colours. In other words, a very small contribution of blue is sufficient to pass from very saturated yellow to white.
The consequence of this is that all red, orange, yellow and a part of the green colours are compressed in the narrow triangle 1W2. The colour triangle of fig. 16 is therefore not particularly suited to giving a *general picture of all colours.*
How can this be improved? That is to say, how can we arrive at another colour triangle not having the two disadvantages mentioned? Is it possible to construct various colour triangles? If we take the standpoint of chapter III we can answer the latter question at once in the affirmative. In fig. 6 both the place of the colours R, G and B and the position and size of the triangle XYZ were chosen arbitrarily. Merely by selecting this triangle large enough, all spectral colours can find a place in it and the occurrence of negative coordinates is therefore avoided. In order to see the possibility of various types of colour triangles from the more exact standpoint adopted in chapter IV, in which colour triangles are derived from a colour space, we must seek in the formulation of the colour space (sections 24, 25) for points where we have acted in any way arbitrarily. After choosing the system of axes of fig. 10b we agreed to mark off the three coordinates B_1 B_2 and B_3 of any colour K along these axes. In other words, we have placed the spectral colours 700, 546.1 and 435.8 mμ (for which only one of the B's differs from 0 in each case) along the axes and, moreover, we have placed these primary colours at equal distances from O when they displayed the same brightness. This allotting of definite places to these spectral lights (with brightness 1) can be called arbitrary: they might just as well have been placed anywhere else in the space.
If we choose another position all colours are then disposed differently over the space (and therefore also over the colour plane). If we keep

the primary colours on the coordinate axes but place the three colours possessing equal brightness at unequal distances from O (in other words, make the scale in which the colours of different brightnesses are distributed along the axes different for the three primary colours), it would be possible in this way to get the white point right in the middle of the colour triangle. In order, however, to avoid negative coordinates in the colour space (and at the same time in the colour plane too) it has been found necessary to allot a place to the spectral colours 700, 546.1 and 435.8 n o t lying on the coordinate axes. Fig. 14 shows straightaway that these spectral colours have to be placed more to the middle of the shaded triangle if all spectral colours are to have a place inside this triangle. In the following chapter we shall see how to choose the position of the spectral colours 700, 546.1 and 435.8 to obtain a satisfactory colour space and colour triangle.

CHAPTER V

The C.I.E. Coordinate system XYZ

§ 30 *Introduction of the XYZ system*

In 1931 the C.I.E. (see section 16), considering the possibilities mentioned at the end of the previous chapter, recommended a new system of colour coordinates based on new measurements by Guild[4]) and Wright[2]).

Cf. the explanations of this system given by Smith[1]), Judd[8]), Ribaud[1]), Escher[1]), Fleury[1]), Richter[2]) etc.

The introduction of this new system, which has since been adopted internationally and is in general use, aimed at establishing uniformity in the specification of colours, for which hitherto various systems were in use, mostly based on the old measurements of König[6]) (1892), sometimes supplemented by those of Abney[1]), Ives[3]), Exner[2]), and others [see also Judd[1]) and Troland[1])]. The introduction of this new system may therefore be compared with the introduction of the concept of brightness based on the international relative luminosity curve (sections 16 and 17). Here too a selection was made from the measurements hitherto effected and a standardised system based on those measurements was invented. The same law applies to colour measurements as to brightness measurements, *i.e.* a method is correct when its results correspond with the values arising from the standardised definitions.

The new system, generally known as the "C.I.E. system 1931", not only adopts the newer measurements but is moreover arranged so as to make practical calculations as simple as possible. In particular the objections summed up in sect. 29 have been eliminated.

We again take three coordinate axes in space, named in this case the X axis, Y axis and Z axis, forming together equal angles. The coordinates, too, which define the place of a point in colour space are indicated by X, Y and Z. As in fig. 10b, the coordinate Y is again the distance of the point to the plane OXZ (measured in the direction parallel to the Y axis) etc.

Further, in order to give the colour space (and at the same time the colour triangle derived from it) all the properties desired, the following

places have been allotted to the spectrum colours 700, 546.1 and 435.8 mμ (with brightness 1) *(fig. 20)*:

$$\begin{aligned} 700 \text{ m}\mu &: X = 2.7689; & Y &= 1; & Z &= 0, \\ 546.1 \text{ m}\mu &: X = 0.38159; & Y &= 1; & Z &= 0.012307, \\ 435.8 \text{ m}\mu &: X = 18.801; & Y &= 1; & Z &= 93.066. \end{aligned} \tag{8}$$

We cannot here go further into the question of how this clever choice was made.

How can the place of any given colour K be determined in the XYZ space? Let the colour K again be reproducible by the mixture of the colours 700, 546.1 and 435.8 mμ with brightnesses B_1 B_2 and B_3. These three components lie on the three lines given in fig. 20. Their place in the XYZ space is given by their XYZ coordinates. These are, as follows directly from (8):

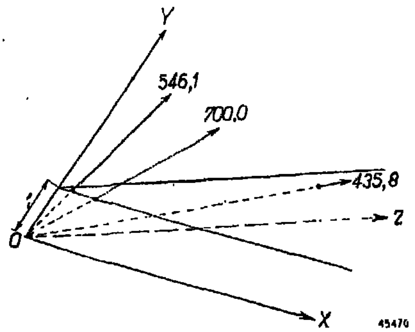


Fig. 20
Introduction of the XYZ system (C.I.E. 1931). The primaries 700, 546.1 and 435.8 (each with brightness 1) are allotted special places in XYZ space. For all three $Y = 1$.

red component

$(\lambda = 700)$: $X = 2.7689\ B_1$; $Y = B_1$; $Z = 0$,

green component

$(\lambda = 546.1)$: $X = 0.38159\ B_2$; $Y = B_2$; $Z = 0.012307\ B_2$,

blue component

$(\lambda = 435.8)$: $X = 18.801\ B_3$; $Y = B_3$; $Z = 93.066\ B_3$.

Now the colour K is obtained by adding these three components. The X, Y and Z coordinates we find (sect. 26) by adding the corresponding coordinates of the separate components: so for the colour K we have:

$$\left\{\begin{aligned} X &= 2.7689\ B_1 + 0.38159\ B_2 + 18.801\ B_3 \\ Y &= B_1 + B_2 + B_3 \\ Z &= 0.012307\ B_2 + 93.066\ B_3 \end{aligned}\right\} \tag{9}$$

These are therefore the coordinates which enable us to calculate the coordinates X, Y, Z of any given colour K when the coordinates B_1 B_2 and B_3 are known. Or, to use a more mathematical term, these are the equations for transforming the B_1, B_2 and B_3 system to the XYZ system.

Now in order to obtain an insight into the distribution of the colours over the XYZ space we must repeat for the XYZ system in a few words all the considerations regarding the $B_1B_2B_3$ system given in chapter IV. With this end in view we shall sometimes use the formulae (9) and sometimes adopt more mathematical considerations, whichever best furthers our desire to keep the reasoning as simple as possible. In many cases it will appear that we can transfer the properties discovered in chapter IV without further ado to the XYZ system. In other cases the properties of the two systems seem to differ considerably.

In the XYZ system each colour has its allotted point. The formulae (9) enable us to determine the coordinates of that point. As a descriptive method the XYZ system is therefore equivalent to the $B_1B_2B_3$ system (cf. sect. 24).

The property of sect. 25: "If we only alter the brightness of a colour, the colour point moves along a straight line through O", is unchanged, for, when the brightness of the colour K is raised a factor n, B_1 B_2 and B_3 will also be increased by the same factor n (sect. 25) and according to (9) the same applies to the quantities XYZ. The quantities therefore increase proportionally, so that in the XYZ space we also proceed along a straight line through O.

In particular we shall now consider the straight lines each representing a certain spectral colour which together form the surface of a cone.

In sect. 25 we established the corresponding straight lines by establishing experimentally the three brightnesses B_1 B_2 and B_3 required to match each spectral colour at one particular brightness. In table B (the tables denoted by a letter are at the end of this book) these experimental results have been reproduced: $\overline{B}_1$ $\overline{B}_2$ and $\overline{B}_3$ give therefore the space coordinates of the spectral colours, presupposing that the power (number of watts) is the same for each colour.

The use of capitals signifies that we are concerned with space-coordinates (see the agreement made in sect. 27), the bar above the letter reminds us that the quantities are related to colours with equal power.

All the tables at the end of the book are based on the measurements of Guild and Wright, mentioned before, adopted by the C.I.E. for the specification of colour.

Of course all values $\overline{B}_1$ $\overline{B}_2$ and $\overline{B}_3$ might have been multiplied by any given equal amount.

If we set out the values of $\overline{B}_1$ $\overline{B}_2$ and $\overline{B}_3$ for each wavelength we

get three curves called by the not very happily chosen name of "distribution curves" (of the spectrum of equal energy) or tristimulus values.

In addition the accessory trichromatic coefficients (coordinates in the colour plane) b_1 b_2 b_3 are given in table B (see sect. 27). It can be seen that almost all spectral colours do indeed possess one negative coefficient.

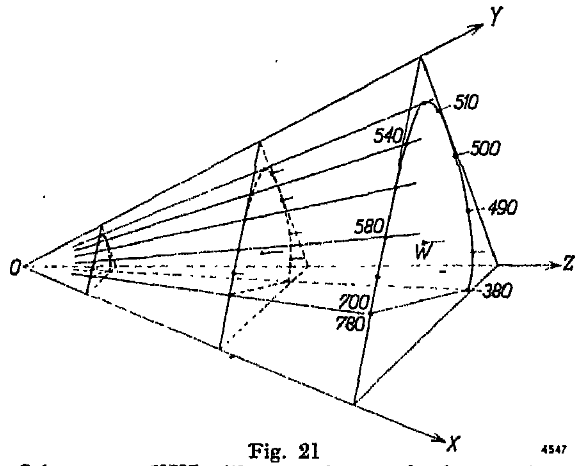


Fig. 21

Colour space XYZ with cone of spectral colours and the intersections of this cone with a number of parallel planes. The lines OX, OY, OZ do not represent chromaticities.

From the coordinates $\overline{B}_1\,\overline{B}_2\,\overline{B}_3$ given in table B we can now calculate the corresponding coordinates $\overline{X}$ $\overline{Y}$ and $\overline{Z}$, according to formula (9). Table D gives the results, which of course again apply to spectral colours of equal powers. Besides $\overline{X}\,\overline{Y}\,\overline{Z}$, the corresponding coefficients, $x = \overline{X} : (\overline{X} + \overline{Y} + \overline{Z})$; $y = \overline{Y} : (\overline{X} + \overline{Y} + \overline{Z})$ and $z = \overline{Z} : (\overline{X} + \overline{Y} + \overline{Z})$ of the colour plane are given (table E).

We see straightaway from table E that one of the objections to the old system has disappeared: there are no longer any negative coordinates. This also appears from *fig. 21*. Here the cone surface of the spectral colours (analogous to fig. 12) as it follows from table D has been drawn, again with some sections with planes cutting off equal parts of the X, Y and Z axis. We see that the whole cone surface — in contradistinction to fig. 12 — lies within the pyramid: this is the necessary condition for the avoidance of negative coordinates.

In figs. 11 and 12 the axes O1, O2 and O3 represented certain kinds of colour, namely those selected as primary colours. In fig. 21 on the other hand the axes OX, OY and OZ lie outside the colour cone and therefore represent no colours. These three directions only serve — just as the triangle XYZ in fig. 6 — to make it possible to give the position of the colour points in numbers.

§ 31 *Further properties of the XYZ system*

Let us now mix two colours K (coord. $B_1 B_2 B_3$) and K′ (coord. $B_1' \, B_2' \, B_3'$). The mixture has, according to sect. 26, the coordinates $B_1 + B_1'$; $B_2 + B_2'$; $B_3 + B_3'$.

The X coordinate of K is therefore

$$2.7689 B_1 + 0.38159 B_2 + 18.801 B_3 = X$$

that of K′:

$$2.7689 B_1' + 0.38159 B_2' + 18.801 B_3' = X'$$

and that of K + K′:

$$2.7689 (B_1 + B_1') + 0.38159 (B_2 + B_2') + 18.801 (B_3 + B_3')$$

From this it follows immediately that the X coordinate of the mixture is equal to $X + X'$. If the same reasoning is applied to the Y coordinate and the Z coordinate we find that the thesis proved in sect. 26: "the coordinates of the mixture are found by adding the corresponding coordinates of the two components" obtains here too. Geometrically this means again that the addition of K and K′ takes place by parallelogram construction (fig. 13). Hence it follows in sect. 26 that mixtures of two or more spectral colours always remain within the cone which is bounded by the spectral colours and the plane rOv. Within this cone each point represents a certain colour, while points outside the cone represent no colour. This proves at the same time that negative coordinates never occur even with composite colours.

The quantities $x = X : (X + Y + Z)$; $y = Y : (X + Y + Z)$ and $z = Z : (X + Y + Z)$ can again be plotted in the colour plane by the two methods of fig. 15a and 15b. *Fig. 22a* gives the colour triangle in the equilateral form. We have already met this method of representation in figures 6 and 7. As only two of the three coordinates are used for the rectangular triangle there are three possibilities *(fig. 22b-d)* of fitting the colours into a rectangular triangle on the C.I.E. system. From practical considerations the use of the rectangular triangles has lately become more and more usual. There is no international agreement regarding the choice of the three possibilities. In practice forms 22b and 22d are met with more or less exclusively (the former particularly in the Netherlands and the latter in England, France, America and Germany). We shall choose for our further illustrations the form 22d. We shall not discuss further the minor advantages and disadvantages of each of the triangular shapes. As the shapes 22b and 22d are both met with in the literature on the subject it is desirable to become familiar with both representa-

tions. For this reason a few figures will be given according to both methods. When the rectangular forms are used the oblique side is usually omitted, as we are only interested in the distances from the colour points to the two sides of the rectangle (see for instance fig. 25). For the sake of convenience we shall continue to speak of a "colour triangle" although actually the expressions "colour dia-

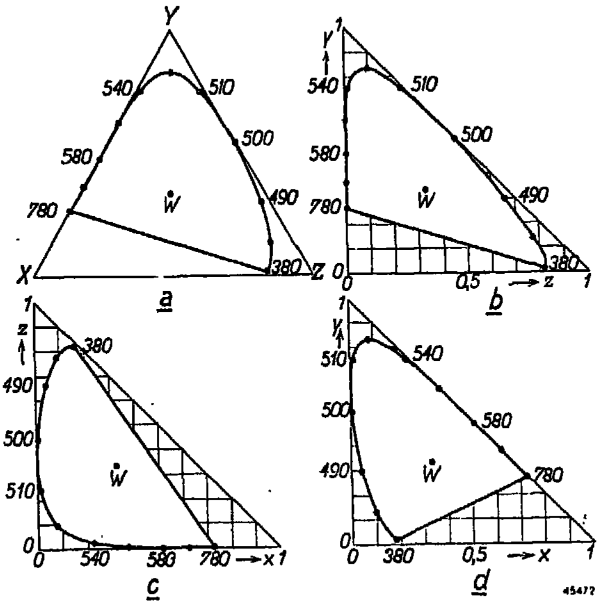


Fig. 22
Colour triangles or chromaticity diagrams derived from the XYZ system. a) equilateral form; b, c, d) rectangular forms.

gram" and "chromaticity diagram" or "chromaticity chart" and the more general "colour plane" are more correct in this case.

Fig. 22a, the colour triangle in equilateral shape, corresponds with the figure occurring in the sections of fig. 21.

By analogy with the corresponding proof of section 27 (fig. 17) we find $h_1 : X = q$; $h_2 : Y = q$ and $h_3 : Z = q$; therefore $h_1 : h_2 : h_3 = X : Y : Z$. The distances from K to the sides of the triangle are therefore in the same proportion as the space coordinates X, Y and Z.

From this property it follows again (see fig. 18) that in the colour plane the mixture k + k′ lies on the connecting line of k and k′. All conclusions drawn in chapter III from this property evidently

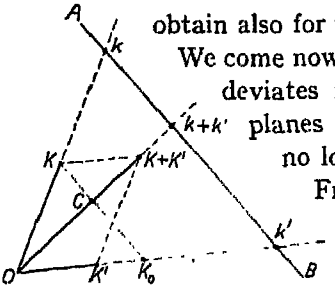


Fig. 23
Derivation of the formula $(k \to k + k') : (k + k' \to k') = \frac{B'}{y'} : \frac{B}{y}$.

obtain also for the colour triangle of the C.I.E. system. We come now to a property in which the XYZ system deviates from the $B_1B_2B_3$ system, namely the planes of the transverse section in fig. 21 are no longer planes of equal brightness.

From the second example of the equations (9) of sect. 30, $Y = B_1 + B_2 + B_3$ it follows that $Y = B$, so that we find for the planes with constant brightness the planes with constant Y, *i.e.* the planes parallel to the coordinate plane XOZ. Fig. 20 illustrates such a plane. For the planes of transverse section in fig. 21, however, $X + Y + Z =$ constant and we see also that they do not run parallel to the plane XOZ, so that these planes are not planes of constant brightness.

Directly connected with this deviating property of the XYZ space is the fact that the last property of sect. 28, the position of the mixture $k + k'$ between k and k′, does not remain unaltered for the XYZ triangle.

In contrast to what was found previously (fig. 18): $(k \to k + k') : (k + k' \to k') = B' : B$ we find for the new system the formula:

$$(k \to k + k') : (k + k' \to k') = \frac{B'}{y'} : \frac{B}{y}.$$

or, verbally, in the new colour plane the distances of the mixed colour $k + k'$ to the two components k and k′ stand as $\frac{B'}{y'} : \frac{B}{y}$, therefore as the quotients of the brightnesses and the Y-coordinates in the colour plane of the two components.

The derivation of this formula is carried out in exactly the same way as that of the corresponding formula sect. 28.

Algebraically: the space coordinates of the two components and their mixture are respectively (XYZ), $(X'Y'Z')$ and $(X''Y''Z'')$, their coördinates in the colour plane are (xyz), $(x'y'z')$ and $(x''y''z'')$ and their brightnesses Y, Y', Y''. Then $(k \to k + k') : (k + k' \to k') = (y'' - y) : (y' - y'')$.

If the equations $y = Y : (X + Y + Z)$; $y' = Y' : (X' + Y' + Z')$; $y'' = Y'' : (X'' + Y'' + Z'')$; $X'' = X + X'$; $Y'' = Y + Y'$ and $Z'' = Z + Z'$ are substituted successively it can be calculated that

$$(k \to k + k') : (k + k' \to k') = (X' + Y' + Z') : (X + Y + Z) = \frac{Y'}{y'} : \frac{Y}{y} = \frac{B'}{y'} : \frac{B}{y}.$$

Geometrically: In *fig. 23* K and K′ again represent the colours to be mixed and AB the intersecting line of the plane of drawing with a surface cutting off equal parts of the axes OX, OY and OZ. We now get for the brightnesses of k and k′ (considered as points of space) : $B_k : B_{k'} = Y_k : Y_{k'} = y_k : y_{k'}$ ($X + Y + Z$ is the same for all points of AB). Likewise $B_K : B_{K_0} = y_k : y_{k'} = y : y'$.
For the ratio $OK' : OK_0$ we get therefore:

$$OK' : OK_0 = B' : B_{K_0} = \frac{B'}{B}\ \frac{y}{y'} = \frac{B'}{y'} : \frac{B}{y}.$$

But now: $(k \to k + k') : (k + k' \to k') = KC : CK_0 =$

$$= (K \to K + K') : OK_0 = OK' : OK_0 = \frac{B'}{y'} : \frac{B}{y}.$$

This property is more complicated than the corresponding property for the $B_1B_2B_3$ system which we derived in sect. 28, but this is an unavoidable consequence of our endeavour to bring the white point more into the centre of the colour triangle. For, if we imagine the reproduction of white by a mixture of the complementary colours yellow and blue, this endeavour means that we wish to modify the position of white in respect to these two components, and it is clear that the simple property of sect. 28 is then lost.

Finally it must be remarked that the theorem just proved for the position of a mixed colour obtains unaltered for the rectangular triangle. In *fig. 24* the same mixture of a colour C from the components A and B is illustrated in the equilateral and rectangular triangle. Seeing that in both representations the distance from a colour point to the base is equal to y, the dotted lines are all parallel, so that

$$\frac{A'C'}{C'B'} = \frac{AC}{CB} = \frac{B_B}{y_B} : \frac{B_A}{y_A}.$$

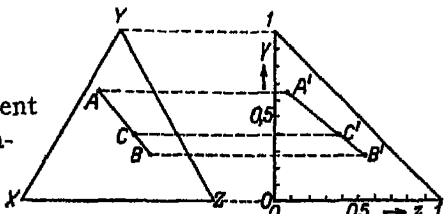


Fig. 24
The formula of fig. 23 also obtains in the rectangular diagram.

We shall now return for a moment to the second example of equations (9) namely: $Y = B_1 + B_2 + B_3$.
From this formula we read that in the C.I.E. system the

Y coordinate is proportional to the brightness. In table D·the column $\overline{Y}$ also represents the brightnesses of spectral colours which all radiate the same power. But these brightnesses are by definition proportional to the international relative luminosity curve V_λ (see sect. 17). We can rightly call this column therefore: $\overline{Y} = V_\lambda$.

§ 32 *General transformations of colour space*

Equations (9) of section 30, which tranformed the $B_1B_2B_3$ system into the XYZ system, form a special case of a general homogeneous linear transformation:

$$\left.\begin{aligned} X'' &= a_{11}X' + a_{12}Y' + a_{13}Z' \\ Y'' &= a_{21}X' + a_{22}Y' + a_{23}Z' \\ Z'' &= a_{31}X' + a_{32}Y' + a_{33}Z' \end{aligned}\right\} \qquad (10)$$

transforming a coordinate system $(X'Y'Z')$ into a system $(X''Y''Z'')$ in which the a's are any given constants. The only thing required of these constants is that the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ does not vanish.

If this condition is fulfilled we can find the inverse transformation by solving (10) which transforms $(X''Y''Z'')$ into $(X'Y'Z')$:

$$\left.\begin{aligned} X' &= b_{11}X'' + b_{12}Y'' + b_{13}Z'' \\ Y' &= b_{21}X'' + b_{22}Y'' + b_{23}Z'' \\ Z' &= b_{31}X'' + b_{32}Y'' + b_{33}Z'' \end{aligned}\right\} \qquad (11)$$

in which the b's are connected with the a's in the well-known manner. A point in the $X''Y''Z''$ space always corresponds with a point in the $X'Y'Z'$ colour space and inversely.

Straight lines and plane surfaces are still straight lines and plane surfaces after transformation and parallel lines remain parallel lines. The following properties, which we have already proved for the $B_1B_2B_3$ and $X\ Y\ Z$ systems, hold for all colour spaces connected with those systems by homogeneous linear transformations:

"When the brightness is changed a colour moves along a straight line through O."

"The spectral colours, together with the saturated purple colours, form the outlines of a cone within which all colours lie."

"Mixing of colours takes place by vector addition."

"We find the corresponding colour triangle in a colour space by central projection."

"In the colour triangle the mixture k + k′ lies in one straight line with k and k′."

The simple proofs we can leave to the reader.

As nine arbitrary constants occur in (10), the number of transformations that can be applied to a given system is ∞^9. This number 9 recurs in various ways.

Geometrically we can consider (10) as a transition to another coordinate system. In this case the number of parameters amounts to 6 (for the establishment of the new axes) plus 3 (choice of units along the three axes) = 9.

Physically (10) may mean another choice of primary colours coupled with a different choice of brightness units for the three primary colours. As the three colours, which are to be placed at a distance 1 from O on the three axes, can always be chosen from the three-dimensional colour manifold, the number of parameters is 3×3. This physical formulation is only correct when the coordinate axes both before and after the transformation actually represent colours (in the XYZ system this is not the case). In general, however, the physical formulation obtains which we have already found in sect. 29 and 30: any given place in space can be assigned to the 3 fixed colours (for instance, the spectral colours 700, 546.1 and 435.8 with brightness 1). This is also the formula by which the C.I.E. established the XYZ system. Number of parameters $3 \times 3 = 9$.

In practice we do not bother about whether a general constant increases all the space coordinates by the same factor. We are therefore actually only interested in the ratios of the 9 coefficients of (10). In this way, for practical purposes, the number of parameters has been reduced to 8.

If the system $(B_1B_2B_3)$ (chapter IV) has been chosen for $(X'Y'Z')$ we find by adding the three equations (11):

$$B = B_1+B_2+B_3 \stackrel{L}{=} (b_{11}+b_{21}+b_{31})X'' + (b_{12}+b_{22}+b_{32})Y'' + \\ + (b_{13}+b_{23}+b_{33})Z''.$$

In general, therefore, we can ascertain that the brightness of a colour is always a linear function of its three coordinates. The coefficients occurring in this linear form, which are called *luminosity coefficients*, can assume any given value. For the $B_1B_2B_3$ and the $X\,Y\,Z$ system they assumed the values (1, 1, 1) and 0, 1, 0) respectively.

§ 33 *General transformations of the colour plane*

If colour *space* is transformed the corresponding colour *plane* is automatically transformed as well.
Since the following holds for coordinates in the colour plane (trichromatic coefficients):

$$x' = X' : (X'+Y'+Z'); \quad y' = Y' : (X'+Y'+Z');$$
$$z' = Z' : (X'+Y'+Z')$$

(and analogously for $x''y''z''$) it follows from (9) that the transformation formulae for the colour plane must be:

$$\left.\begin{aligned} x'' &= (a_{11}x' + a_{12}y' + a_{13}z') : \Sigma \\ y'' &= (a_{21}x' + a_{22}y' + a_{23}z') : \Sigma \\ z'' &= (a_{31}x' + a_{32}y' + a_{33}z') : \Sigma \end{aligned}\right\} \quad (12)$$

in which

$$\Sigma = (a_{11} + a_{21} + a_{31})x' + (a_{12} + a_{22} + a_{32})y' + (a_{13} + a_{23} + a_{33})z'$$

From (11) the inverse transformation can be deduced analogously. The transformation (10) of colour space was homogeneous and linear, in other words, it was an *affine* transformation. The transformation (12) of the corresponding colour plane is on the other hand a *projective* transformation.
In consequence of this the connection between the two colour triangles is much less obvious than the connection between the two colour spaces. That is why in the foregoing all properties of the colour plane were deduced from the simpler properties of colour space.
Some of the consequences of the difference between affine and projective transformation are the following:

a. Whereas in the transformation of colour space the plane at infinity passes over again into the plane at infinity, in the transformation of the colour plane the straight line at infinity in general passes over into a finite straight line, while inversely in the $x'y'z'$ plane a finite straight line can in general be indicated, which after transformation plays the part in the $x''y''z''$ plane of the straight line at infinity. The latter we can represent as appears in (11), by $\Sigma = 0$.
   In the transformation of the $B_1B_2B_3$ system to the $X\,Y\,Z$ system, dealt with previously, the straight line at infinity passed over into the straight line $y = 0$.
b. Whereas in the transformation of colour space parallel lines remain parallel, and therefore a parellelogram remains a parallelo-

gram, we can transpose a parallelogram in the colour plane into a quadrangle of any given shape.

c. In the colour plane a projective series of points corresponds with a series of equidistant points in space (for instance, the colours K, 0.9K + 0.1K′, 0.8K + 0.2K′....K′) which remains equidistant in any transformation. This fact explains why, in the colour triangle, in general the mixture of two colours of the same-luminosity is *not* situated half-way between the two components.

In general we shall therefore have to determine anew the position of the mixture of two colours on the connecting line of the components. If the coordinates of the two components are $(X'_{K'}, Y'_{K'}, Z'_{K'})$ and (X'_K, Y'_K, Z'_K) we find, in the same way as in sect. 31: $(\mathrm{k} \to \mathrm{k} + \mathrm{k}') : (\mathrm{k} + \mathrm{k}' \to \mathrm{k}') = (X'_{K'} + Y'_{K'} + Z'_{K'}) : (X'_K + Y'_K + Z'_K)$.
But now $B = L_1X' + L_2Y' + L_3Z'$ holds, wherein $L_1\, L_2\, L_3$ are the luminosity coefficients (sect. 32), so that we can also write:

$$(\mathrm{k} \to \mathrm{k} + \mathrm{k}') : (\mathrm{k} + \mathrm{k}' \to \mathrm{k}') = \frac{B_{K'}}{L_1x'_{K'} + L_2y'_{K'} + L_3z'_{K'}} : \frac{B_K}{L_1x'_K + L_2y'_K + L_3z'_K}$$

For $L_1 = L_2 = L_3 = 1$ this property passes into that of sect. 28, for $L_1 = 1$, $L_2 = 0$ and $L_3 = 1$ into that of sect. 31.
Finally we notice that the distance of k + k′ to k and k′ only stands as $B_{K'} : B_K$ when k and k′ lie on a straight line $L_1x' + L_2y' + L_3z' =$ constant. In the C.I.E. system these are the straight lines $y =$ constant. In the $B_1B_2B_3$ system all straight lines conform to the requirement that $L_1b_1 + L_2b_2 + L_3b_3 = b_1 + b_2 + b_3 = 1$ and the simple property with regard to the position of k + k′ therefore holds.

§ 33a *Quantity of a colour; trichromatic unit*

* In section 25 attention has been drawn to the fact that two colours K (coord. X', Y', Z') and nK (coordinates nX', nY', nZ'), belonging to the same colour species, are sometimes said to represent different quantities of the same colour (or colour species), the quantities being in the proportion $1 : n$.

As a "trichromatic unit" of the colour K (having the trichromatic coefficients or coordinates in the colour *plane* $x'_K\, y'_K\, z'_K$) one defines the colour whose coordinates in colour *space* are x'_K y'_K and z'_K. The quantity of a colour K $(X'Y'Z')$ is therefore expressed in trichromatic units by
$X'_K : x'_K = Y'_K : y'_K = Z'_K : z'_K = (X'_K + Y'_K + Z'_K) : (x'_K + y'_K + z'_K) = (X'_K + Y'_K + Z'_K) : 1 = X'_K + Y'_K + Z'_K$.
Denoting $X'_K + Y'_K + Z'_K$ by m_K and $X'_{K'} + Y'_{K'} + Z'_{K'}$ by m_K' it is seen from section 33 that in the colour *plane* the colour point of *the mixture* k + k′ *is determined by*

$$(\mathrm{k} \to \mathrm{k} + \mathrm{k}') : (\mathrm{k} + \mathrm{k}' \to \mathrm{k}') = m_{K'} : m_K.$$

In words: in the colour plane the distances of the colour point of the mixture to the colour points of the components are inversely proportional to the quantities of the colours K and K′ expressed in trichromatic units of each colour (see section 69).

The relation of the unit colour (k) to the primaries of $X'Y'Z'$ space may also be expressed by the colour equation

$$(\mathrm{k}) \longleftrightarrow x'_{\mathrm{K}}\ (\mathrm{X}') + y'_{\mathrm{K}}\ (\mathrm{Y}') + z'_{\mathrm{K}}\ (\mathrm{Z}'),$$

or in words: one trichromatic unit (k) of the colour K matches the additive mixture of x'_{K} units of (X′), y'_{K} units of (Y′) and z'_{K} units of (Z′). The colour (X′), being one trichromatic unit of the X′ primary, is represented by the coordinates $X' = 1$, $Y' = Z' = 0$ etc.

The coefficients of the colour equation adding up to 1, such an equation is sometimes denoted as a "unit equation". Some authors prefer to build up the theory of the transformation of colour space by means of unit equations [see W r i g h t[10]) and section 36a].

§ 34 *Requirements a transformation should fulfil*

Since the existence of the ideas of colour space and colour triangle all kinds of desires have been expressed regarding their properties. We can satisfy some of these requirements by passing over with the aid of a skilfully chosen transformation to a system possessing the desired properties. This "skilful choice" of the transformation consists always of a suitable selection of the 8 ratios of the coefficients of transformation (10). As we have not less than 8 arbitrary parameters at our disposal the possibilities for fulfilling requirements are very great. This fact is continually being lost sight of by some authors. A striking example is the application, by H e l m h o l t z[2]) of K ö n i g's[2]) measurements of sensitivity to wavelength differences to a general theory, based on an extension of the Weber law (see sect. 77). In this application H e l m h o l t z had so many parameters at his disposal, which he could vary *ad lib.*, that every mathematician would have been astonished if this application (to a fairly simple experimental curve) had not succeeded with reasonable accuracy. And yet P e d d i e[1]) and A l l e n[1]) forty years later still spoke of the success of that application as "one of the strongest and most direct evidences for the establishment of the trichromatic theory of vision" (see also sections 80 and 81). All the same there are of course a great number of requirements that cannot possibly be fulfilled.

a. Requirements in conflict with either the affine or the projective character of the transformation. Thus, for instance, it is impossible to transform the curve of the spectral colours to a circle and thus to imitate the colour circle of N e w t o n[1]). This would only be

possible if this curve in the XYZ triangle was already a conic section.

b. Requirements which are mutually conflicting. See the example mentioned in section 31. It is impossible to place the white point about equally distant from all spectral colours and for the property that the mixture of two colours of equal brightness always lies half way between the two components to obtain at the same time. According to sect. 33 the latter requirement is equivalent to the equality of the three luminosity coefficients.

c. A combination of requirements representing more than 8 independent conditions.

We shall now give two examples of combinations of requirements which can be fulfilled. In earlier systems [K ö n i g [6])] very special directions have been chosen as axes in colour space (the isochrome directions of the dichromats, see sect. 65) and moreover daylight was placed in the centre of gravity of the colour triangle. The fulfilment of the first requirement makes six independent demands on the parameters; and the second requirement two. The 8 degrees of freedom therefore exactly allow of the fulfilment of all these requirements [see also S c h r ö d i n g e r [2])]. The combination of demands made on the XYZ system of the C.I.E. is a little more complicated. In the first place the equal energy spectrum must lie at the centre of gravity of the colour triangle. This amounts to two independent demands.

In the second place we have the simplest possible luminosity formula $B = Y$ (the possibility $L_1 = L_2 = L_3$ was already ruled out by the first requirement!). Two requirements.

In the third place the plane $X = 0$ must touch the cone surface of the spectral colours along the generator represented by $\lambda = 507$ mμ (see fig. 21). Two requirements.

Finally the plane $Z = 0$ must touch the said cone as well, and that along the generator representing $\lambda = 700$ mμ. This makes another two requirements.

The requirements regarding the contact with the planes $X = 0$ and $Z = 0$ were for the purpose of causing the colour triangle to enclose the curve of the spectral colours as closely as possible. In this second example also eight separate conditions have been expressed. Another requirement sometimes asked of a projective transformation is the preservation of the shape of the curve of the spectral colours. If we require this curve to remain unaltered both as regards shape and dimensions, there only remain three degrees of freedom (one

rotation and two translations in the colour plane). This requirement appears therefore to have been five-fold. If we only require the *shape* to be preserved a four-fold demand has been made which we can formulate as follows:

a. The triangle formed by any three spectral colours must keep its shape (two-fold condition).
b. The relative position of the white point with regard to this triangle remains unchanged (two-fold condition).

The transformations of the colour plane conforming to these requirements are affine (the straight line at infinity keeps its place) and form a sub-group of the group of general projective transformations. For literature regarding transformations of the colour plane and space see Ives[3, 6]), Schrödinger[2, 7]), Froehlich[1]), Guild[1]), Dziobek[1]), Judd[3]), MacAdam[2]) etc.

§ 35 *The monochromatic system*

Besides the trichromatic system with which we have been chiefly occupied so far, and which is based on the possibility of matching each colour by mixing three primary colours, there is another system in use, namely the monochromatic system, based on the possibility of reproducing each colour by mixing a "white" colour and a spectral colour (see also sect. 22).
Instead of the three coordinates XYZ of the trichromatic system we now have the three quantities: dominant wavelength (λ_d), colorimetric purity (p) and luminosity (B). These conceptions have already been defined in sect. 22.
One may perhaps wonder why this system has also been maintained beside the entirely adequate XYZ system, and why it was considered worth while continually to translate the values of the one system into that of the other.
In a certain sense tradition has played a part in this. Historically the monochromatic system can boast of the same respectable old age as the trichromatic system, and both systems have been maintained and developed side by side in the course of time.
But this historic development would not have been possible if the monochromatic system had not possessed some advantages lacking in the other system. These advantages are as follows:

a. For people not steeped in the subject the exact significance of the monochromatic coordinates is much easier to understand than that of the trichromatic coordinates. This is the more so because

at the very beginning of the development of the trichromatic system coordinate axes were adopted which lay outside the cone-shaped area of the colours (see for example fig. 21), while a less happily chosen terminology made the subject still more inaccessible for the layman.

b. The definition of the monochromatic coordinates is less arbitrary than that of the trichromatic coordinates. For a long time very different trichromatic systems were in use side by side and it was almost an impossible task to show adequate accurate relationships between the different systems. But in the use of the monochromatic system far greater unity existed; the mutual differences consisted chiefly in a slightly divergent choice of white. Since, however, thanks to the C.I.E., by far the greater part of humanity makes use of the same XYZ system (or a system which can be directly calculated back into it), this advantage of the monochromatic system has practically disappeared.

c. The most important advantage of the monochromatic system is considered to be the fact that it is in practice easier to get a general picture from it than from the trichromatic system. Indeed if the trichromatic coefficients xyz of a colour are given it is practically impossible to form from this an approximate idea of the character of the colour. If, however, the monochromatic coordinates are given such an idea can easily be made from this, *e.g.* "saturated bluish green", "pale orange" etc.

The advantages are very obvious if two colours differing only slightly are to be compared. Yet this advantage is not so great as appears at first sight. We must bear in mind that the colour sensation depends very largely on the surroundings and the condition of the observer's retina (influences which we eliminated as far as possible in the case of colour measurement). Thus the monochromatic coordinates under certain circumstances can even cause an incorrect representation of the colour impression.

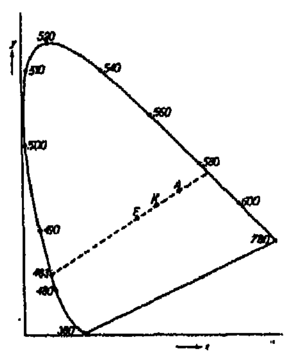


Fig. 25
The chromaticity K may evoke the sensation of pale yellow (λ_d = 583mμ) or of pale blue (λ_d = 483 mμ) according as the surroundings are illuminated by E or A.

Fig. 25 illustrates such a case. E and A represent respectively the colour points of standard white E and incandescent lamp light, and K a colour situated between the two. If E is used for the white point for

the monochromatic system we find for the dominant wavelength of K the value $\lambda = 583$ mμ (yellow), so that it might be expected that K would make a p a l e y e l l o w impression under all circumstances. This is, however, not the case. There are circumstances in which K makes quite a different impression. If, for instance, we look at a large white surface illuminated by incandescent light and allow a small object to appear in the field of vision, which with this illumination possesses the colour coordinates of K, the object will then look p a l e b l u e. The idea evoked by the value of λ appears therefore in this case to be incorrect.

Such gross contradictions can only be disposed of by choosing another white point each time in accordance with the illumination used. If, in the case described above, we choose A as white point for the determination of the monochromatic coordinates, we find for K the value $\lambda = 483$ mμ (blue). This value evokes the correct idea that we have here a pale blue colour impression.

Only when the surroundings have a colour whose coordinates do not deviate too strongly from those of E, and the observed colour sensation is not too unsaturated, the monochromatic coordinates (starting from white point E) can evoke a fairly correct idea of the nature of the sensation.

Here we see, therefore, the following disadvantage of the monochromatic system emerging. If we wish to profit to the utmost from advantage (c) we must every time choose a new white point, so that one particular set of monochromatic coordinates no longer belongs to a particular point in the colour surface. The monochromatic coordinates no longer follow unambiguously from the trichromatic ones.

A second disadvantage is that the monochromatic system does not lend itself to making calculations. While we are able to give simple instructions about the mixing of colours and the calculation of a compound colour for the trichromatic system, this is not possible for the monochromatic system. The only way to calculate a mixed colour from its components given by λ_d, p and B is to calculate the trichromatic coordinates for the various components, to apply the well-known mixture laws and finally to determine the λ_d, p and B again from the results.

Opinions may differ, after considering the various advantages and disadvantages, as to whether it is desirable to maintain the monochromatic system, but everyone will agree that it is certainly n o t desirable to support t w o monochromatic systems. And yet this is,

we are sorry to say, a fact. Besides the conception colorimetric purity defined in sect. 22, there is yet another conception of purity in use, indicated by σ *). This is a ratio of distances in the X Y Z colour triangle, namely (fig. 7) for purple colours the relation WK' : WA, for the other colours WK : WS.
As in the case of p, σ also increases from 0 to 1 when the colour moves from the white point to the spectral curve, but in a different manner from p (see sect. 40).
In the long run one of the two quantities p and σ will disappear, but it cannot yet be foretold which will survive. Personally I should prefer to drop σ, as this magnitude has something artificial about it, and is, moreover, connected with the — more or less arbitrarily chosen — XYZ system. But in various countries the opposite tendency exists. For distinguishing between the two conceptions it will in many cases be sufficient to use the letters p and σ. If it is desired to distinguish the conceptions by name, p should be called "colorimetric purity" and σ "excitation purity".

*) σ (pronounce sigma) is the Greek letter s.

CHAPTER VI

Colour calculations in the C.I.E. system

§ 36 *Calculation of the colour coordinates of a light source*

The most frequently recurring problem of calculation is as follows: determine the colour coordinates of a mixture when the components are given. This problem takes various shapes according to the way in which the components are given. Finally, however, the calculation is always based on the property mentioned in sect. 31, *i.e.* "the coordinates XYZ of a mixture are found by adding the corresponding coordinates of the components."

If the components in the $X\,Y\,Z$ space possess the coordinates $(X_1Y_1Z_1)$, $(X_2Y_2Z_2)$, $(X_3Y_3Z_3)$ ....

then it follows that

$$\begin{aligned} X &= X_1 + X_2 + X_3 \ \ldots\ldots \\ Y &= Y_1 + Y_2 + Y_3 \ \ldots\ldots \\ Z &= Z_1 + Z_2 + Z_3 \ \ldots\ldots \end{aligned} \tag{13}$$

The calculation is simplest, of course, when we are dealing with a line spectrum and the coordinates of the components are directly given.

As an example we take the light of a mercury lamp in which the mercury vapour has a pressure of 1 atmosphere [for instance the mercury lamps of the type MA manufactured by Philips. See for further particulars U y t e r h o e v e n[1])].

λ	colour	X_λ	Y_λ	Z_λ
578	yellow	31.20	31.28	0.06
546.1	green	10.74	28.14	0.36
435.8	blue	7.33	0.39	36.30
407.8	violet	0.08	0.00	0.40
404.7	violet	0.27	0.01	1.26

This light possesses the components given in the above table.

If we add the contributions of the separate components according to formula (13) we get the following for mercury light:

$$X = 49.62; \quad Y = 59.82; \quad Z = 38.38.$$

As $X + Y + Z = 147.82$ we find for the trichromatic coefficients (coordinates in the XYZ triangle):

$$\begin{aligned} x &= 49.62 : 147.82 = 0.336 \\ y &= 59.82 : 147.82 = 0.405 \\ z &= 38.38 : 147.82 = 0.2595 \end{aligned}$$

In *fig. 26* (C.I.E. triangle in the shape of fig. 22d) this result has been included in the diagram: the point Hg represents the colour of this mercury light.

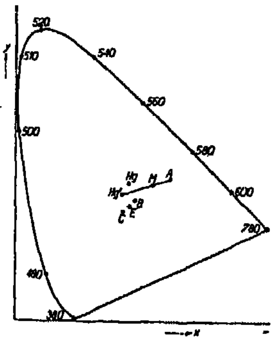


Fig. 26
Calculated colour points of a number of light sources plotted in the XYZ triangle A, B, C, standard illuminants. E = equal energy white. Hg, Hg' two kinds of mercury light. M = Blended light (mixture of Hg' and A).

As a second example let us calculate the colour point of the "equal energy spectrum" (white point E, see sect. 23).

This is a continuous spectrum; therefore we must divide the spectrum (as in sect. 14 and 17) into a large number of equal intervals and add the contributions from each interval. Now table D, mentioned already in sect. 30, contains the XYZ values for the spectral colours when all have the same power; exactly the contributions required here. We therefore find the XYZ of the white point E by adding the coordinates of table D. The result is as follows: $X = 21.37$, $Y = 21.37$, $Z = 21.36$. From this we find the coordinates in the colour triangle, as follows:

$$x = 0.3334; \quad y = 0.3334; \quad z = 0.3332.$$

The white point E therefore lies exactly at the centre of gravity of the triangle. This is of course no coincidence. The XYZ system was previously selected so as to comply with this condition.

We can also check the results we have just found in the following way. By adding the space coordinates $(\overline{B}_1\overline{B}_2\overline{B}_3)$ of table B we find the space coordinates of the white point E in the $B_1B_2B_3$ system:

$$B_1 = 1.891 \qquad B_2 = 8.681 \qquad B_3 = 0.1136.$$

Hence we calculate according to equation (9) of sect. 30:

$$X = 10.685 \qquad Y = 10.685 \qquad Z = 10.68 \text{ or}$$
$$x = 0.3334 \qquad y = 0.3334 \qquad z = 0.3332.$$

Beside the XYZ system — intended for general use — the C.I.E. has introduced another system (RGB) for the special purpose of establishing any possible new measurements regarding colour coordinates of spectral colours. This coordinate system lies between the $B_1B_2B_3$ system and the $X\ Y\ Z$ system. The same choice has been made as regards coordinate axes as in the $B_1B_2B_3$ system, namely the spectral colours 700, 546.1 and 435.8 m, but care has been taken that the white point E (as in the $X\ Y\ Z$ system) falls at the centre of gravity of the triangle. By these qualifications the system is com-

pletely established, as follows: $3 \times 2 + 2 = 8$ (see section 34). The relation between $(B_1B_2B_3)$ and (RGB) is simply:

$$R = B_1 \quad ; \quad G = 0.21783\ B_2 \quad ; \quad B = 16.639\ B_3.$$

The relation between (RGB) and (XYZ) is as follows:

$$X = 2.7689\ R + 1.7518\ G + 1.1302\ B,$$
$$Y = R + 4.5909\ G + 0.06012\ B,$$
$$Z = 0.05650\ G + 5.5944\ B.$$

Table C gives the spacial coordinates $(\overline{R}\,\overline{G}\,\overline{B})$ of the spectral colours of equal power and the accessory trichromatic coefficients (rgb). Here again negative values occur. The addition of all $\overline{R}$ $\overline{G}$ and $\overline{B}$ values of table C gives for the white point E:

$R = 1.891,\qquad G = 1.891,\qquad B = 1.889$ and therefore
$r = 0.3334,\qquad g = 0.3334,\qquad b = 0.3332.$

From $Y = R + 4.5909G + 0.06012B$ it follows directly that the luminosity coefficients of the RGB system are as follows:

$$L_1 = 1 \quad , \quad L_2 = 4.591 \quad , \quad L_3 = 0.0601.$$

§ 36a *The transformation of RGB into XYZ*

* As an example of what has been said in sections 32, 33 and 33a we shall derive the transformation equations connecting the C.I.E. systems (RGB) and (XYZ).

The trichromatic coefficients of the primaries R, G, B are:

	x	y	z	
(R) (λ = 700 mμ)	0.73467	0.26533	0	
(G) (λ = 546.1 mμ)	0.27376	0.71741	0.00883	(a)
(B) (λ = 435.8 mμ)	0.16658	0.00886	0.82456	

as may be derived from eq. (8), x, y and z representing the coordinates of one trichromatic unit of (R), (G) and (B) in the XYZ colour space. When the colour points represented by (a) are transformed to the RGB space the corresponding colour points lie on the R, G and B axis respectively, but they will not represent trichromatic units in the new system. The coordinates in the RGB space therefore will be

	R	G	B	
(R) (λ = 700 mμ)	α	0	0	
(G) (λ = 546.1 mμ)	0	β	0	(b)
(B) (λ = 435.8 mμ)	0	0	γ	

The transformation between the two systems is given by eqs (10), where $X''Y''Z''$ and $X'Y'Z'$ are to be replaced by XYZ and RGB respectively; substituting the values from (b) into (a) we find

$$0.73467 = a_{11}\alpha \qquad 0.26533 = a_{21}\alpha \qquad 0 = a_{31}\alpha$$
$$0.27376 = a_{12}\beta \qquad 0.71741 = a_{22}\beta \qquad 0.00883 = a_{32}\beta$$
$$0.16658 = a_{13}\gamma \qquad 0.00886 = a_{23}\gamma \qquad 0.82456 = a_{33}\gamma$$

eqs (10) therefore become

$$\begin{aligned} X &= 0.73467\ \alpha^{-1}\ R + 0.27376\ \beta^{-1}\ G + 0.16658\ \gamma^{-1}\ B \\ Y &= 0.26533\ \alpha^{-1}\ R + 0.71741\ \beta^{-1}\ G + 0.00886\ \gamma^{-1}\ B \\ Z &= \qquad\qquad\qquad\quad 0.00883\ \beta^{-1}\ G + 0.82456\ \gamma^{-1}\ B \end{aligned} \tag{c}$$

To fix the constants α, β and γ we claim that in both systems the standard white E must have equal coordinates and that the colour point representing one trichromatic unit of E in the XYZ system will be transformed to the colour point representing one trichromatic unit of E in the RGB system. The coordinates of one trichromatic unit of E are therefore

$$X = \tfrac{1}{3} \qquad Y = \tfrac{1}{3} \qquad Z = \tfrac{1}{3} \text{ and}$$
$$R = \tfrac{1}{3} \qquad G = \tfrac{1}{3} \qquad \text{and } B = \tfrac{1}{3}$$

and these two points should correspond. Substitution of these values into (c) leads to three linear equations for α^{-1}, β^{-1} and γ^{-1}, from which we find

$$\alpha^{-1} = 0.66694, \qquad \beta^{-1} = 1.1324 \qquad \gamma^{-1} = 1.2006$$

With these values (c) becomes

$$\begin{aligned} X &= 0.48989\ R + 0.31001\ G + 0.20000\ B \\ Y &= 0.17696\ R + 0.81240\ G + 0.01064\ B \\ Z &= \qquad\qquad\quad 0.01000\ G + 0.98999\ B \end{aligned} \tag{d}$$

The coefficients in each of the eqs (d) add up to 1, as should be the case, for substitution of $R = G = B = \frac{1}{3}$ must give $X = Y = Z = \frac{1}{3}$.
The transformation of the xyz colour plane to the rgb colour plane according to eq. (12) is

$$\begin{aligned} x &= \frac{0.48989\ r + 0.31001\ g + 0.20000\ b}{0.66394\ r + 1.13241\ g + 1.20063\ b} \\ y &= \frac{0.17696\ r + 0.81240\ g + 0.01064\ b}{0.66394\ r + 1.13241\ g + 1.20063\ b} \\ z &= \frac{0.01000\ g + 0.98999\ b}{0.66394\ r + 1.13241\ g + 1.20063\ b} \end{aligned} \tag{e}$$

Dividing all coefficients of (d) by the factor 0.17696 we have:

$$\begin{aligned} X &= 2.7689\ R + 1.75189\ G + 1.1302\ B \\ Y &= 1.0000\ R + 4.5909\ G + 0.06012\ B \\ Z &= \qquad\qquad\ 0.0565\ G + 5.5944\ B \end{aligned} \tag{f}$$

These equations again represent a transformation from XYZ to RGB but now one trichromatic unit of E in XYZ is no longer transformed into one trichromatic unit of E in RGB space, as may be seen from the substitution $R = G = B = \frac{1}{3}$ in (f), giving:

$$X = Y = Z = 5.65921 : 3 = 1.8862.$$

However, (f) is the most common form for this transformation. From (f) we see that the luminosity factors in the RGB system are

$$L_1 = 1, \qquad L_2 = 4.5909, \qquad L_3 = 0.06012$$

which values agree with those given in section 36.
From (d) we may finally calculate the reverse transformation

$$\begin{aligned} R &= 2.3647\ X - 0.89656\ Y - 0.46808\ Z \\ G &= -0.51515\ X + 1.4264\ Y + 0.08874\ Z \\ B &= 0.00520\ X - 0.01441\ Y + 1.00921\ Z \end{aligned} \tag{g}$$

whose coefficients again add up to 1 in each horizontal row.
The same problem may be solved by means of colour equations [W r i g h t [10, 11])]. We denote one trichromatic unit of the RGB primaries in the XYZ system by (R), (G) and (B) and in the RGB system by [R],

[G] and [B]. (R) denotes a (spectral) colour (λ = 700 mμ) whose coordinates in XYZ space are given by (a). On the other hand [R] denotes a different "quantity" of the same colour, whose coordinates in RGB space are (100). From (b) it may be seen that (R) = α[R]. The colour (R) is matched by certain quantities of the "primaries" (X) (Y) (Z). It is true that, as has been stressed in sections 20 and 26, (X), (Y) and (Z) are not "real" colours, but for the moment this will be disregarded. The quantities of (X) (Y) and (Z) needed to match (R) are given by (a). So we may write

$$\begin{aligned}
(R) &= \alpha\,[R] \longleftrightarrow 0.73467\ (X) + 0.26533\ (Y),\\
(G) &= \beta\,[G] \longleftrightarrow 0.27376\ (X) + 0.71741\ (Y) + 0.00883\ (Z),\\
(B) &= \gamma\,[B] \longleftrightarrow 0.16658\ (X) + 0.00886\ (Y) + 0.82456\ (Z).
\end{aligned} \tag{h}$$

As before, the problem is to fix the values of α, β and γ. This may be done using the equality of the colours (E) and [E], each representing one trichromatic unit of the standard white in the XYZ and RGB system respectively, where

$$(E) \longleftrightarrow 0.3333\ (X) + 0.3333\ (Y) + 0.3333\ (Z), \tag{i1}$$

whereas [E] will fulfil the requirement

$$[E] \longleftrightarrow 0.3333\ [R] + 0.3333\ [G] + 0.3333\ [B]. \tag{i2}$$

Adding the colour equations (h) after dividing them by α, β and γ (this is allowed in consequence of G r a s s m a n n's laws), the first member is [R] + [G] + [B] and the second member a linear form in (X) (Y) and (Z). From the equality of (i1) and (i2) we see that this linear form must be identical with (X) + (Y) + (Z). This condition leads to three equations in α^{-1}, β^{-1} and γ^{-1}, from which we find again

$$\alpha^{-1} = 0.66694,\ \beta^{-1} = 1.1324 \text{ and } \gamma^{-1} = 1.2006.$$

The transformations (h) (relations between the primary units of both systems) become therefore

$$\begin{aligned}
[R] &\longleftrightarrow 0.48998\ (X) + 0.17696\ (Y),\\
[G] &\longleftrightarrow 0.31001\ (X) + 0.81240\ (Y) + 0.01000\ (Z),\\
[B] &\longleftrightarrow 0.20000\ (X) + 0.01064\ (Y) + 0.98999\ (Z).
\end{aligned} \tag{j}$$

and the inverse transformations

$$\begin{aligned}
(X) &\longleftrightarrow 2.3647\ [R] - 0.51515\ [G] + 0.00520\ [B],\\
(Y) &\longleftrightarrow -0.89656\ [R] + 1.4264\ [G] - 0.01441\ [B],\\
(Z) &\longleftrightarrow -0.46808\ [R] + 0.08874\ [G] + 1.00921\ [B].
\end{aligned} \tag{k}$$

When comparing (j) and (k) with (d) and (g) we see directly how we can derive the colour equations from the corresponding algebraic equations by transposition of the coefficients. The difference between both types of equations proves the necessity of making a clear distinction between colour equations and algebraic equations. We have avoided the danger of confounding them by using the symbol $\longleftrightarrow$ for the colour equations. That danger of confusion really exists is proved by B o u m a [13]), from whose paper this section has been rewritten. In the opinion of B o u m a colour equations should be used as little as possible, e.g. only when deriving the properties of colour space from G r a s s m a n n's laws.

§ 37 *Calculation of the colour coordinates of a light source (continued)*

The hypothetical case given in sect. 36, namely that in which the colour coordinates of the various components are all cut and dried,

seldom occurs in practice. Usually the coordinates have first to be calculated and only then can the colour of the mixed light be determined by addition.

The components of which the light consists may be given in various ways.

a. If the components are the spectral colours of which a heterochromatic source is built up, these are mostly given by the *power* they emit.
b. If the components are the lights from various sources that together form the composite illumination, we usually know the colour points of the separate components and the ratios of the quantities of light radiated by the various sources. The colour of the blended light is the colour produced by additive mixing of the colours of the individual components in proportions corresponding to the intensity ratios of the two sources.
c. When we deal with a light source otherwise unknown to us we must not only make a calculation to determine the colour, but also take measurements.

In case (a) we make use of table D. This gives the colour coordinates of the spectral colours of equal power. Now, seeing that the coordinates increase proportionally as the power increases, we find the colour coordinates of the spectral components by multiplying, for each wavelength occuring in the light species, the numbers of table D by the power with which the wavelength concerned occurs. The space coordinates of the heterochromatic colour we find by adding those of the spectral colours.

As a first example let us calculate once more the colour point of the mercury lamp mentioned already in sect. 36, but we now imagine the composition of the light given by the ratio of the powers E_1 E_2 E_3.... which every wavelength contains. The value X_1 (first coordinate of the yellow component $\lambda = 578$ mμ) we calculate by multiplying E_1 by the value $\overline{X}(578)$ which we find by interpolation from table D:

λ	colour	E_λ	$X_\lambda = E_\lambda \overline{X}_\lambda$	$Y_\lambda = E_\lambda \overline{Y}_\lambda$	$Z_\lambda = E_\lambda \overline{Z}_\lambda$
578	yellow	35·2	31·20	31·28	0·06
546·1	green	28·6	10·74	28·14	0·36
435·8	blue	22·0	7·33	0·39	36·30
407·8	violet	2·4	0·08	0·00	0·40
404·7	violet	11·8	0·27	0·01	1·25

After addition we find:

$$X = 49.62; \quad Y = 59.82; \quad Z = 38.38$$

or $$x = 0.336; \quad y = 0.405; \quad z = 0.2595.$$

An analogous calculation for the light of another type of mercury lamp (the so-called super high pressure mercury lamp, *viz.* a lamp of the MF type, mercury pressure about 20 atmospheres, manufactured by Philips) gives:

$$x = 0.309; \quad y = 0.367; \quad z = 0.324 \text{ (Hg}' \text{ in fig. 26).}$$

As a second example let us calculate the colour point of the standard white A (see sect. 23), of which the relative spectral distribution is given in table F, namely E_A. Now for each wavelength E_A must be multiplied by the respective $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ from table D. This calculation has been worked out in table G. A series of numbers, as in table G (obtained by multiplying the radiated power of a given light source for each wavelength by the $\overline{X}\,\overline{Y}\,\overline{Z}$ given in table D) is called a trichromatic distribution curve. Thus, for example, tables G, H and I give respectively the trichromatic distribution curves of standard whites A, B and C, while the values of table D are themselves to be taken as the trichromatic distribution curves or tristimulus values of the equal energy spectrum (see also sect. 30).
Addition of the values of table G gives us coordinates of the normal white A:

$$X = 1185; \quad Y = 1079; \quad Z = 383.4,$$

or the triangular coordinates:

$$x = 0.4475; \quad y = 0.4075; \quad z = 0.145.$$

The latter are plotted in fig. 26 (point A).
In the same way tables H and I give the calculation of the colour points of standard illuminants B and C, the spectral distributions of which are also given in table F. We find

for B: $x = 0.3485; \quad y = 0.352; \quad z = 0.2995,$
for C: $x = 0.310; \quad y = 0.3165; \quad z = 0.3735.$

The points in the colour triangle are again plotted in fig. 26. As an example of case (b) we shall blend the mercury light Hg′ of fig. 26 (mercury lamp of the ME type) with incandescent light, and that in the ratio of 1 : 2, as regards the light quantities. For the mercury light we found $x_1 = 0.309$; $y_1 = 0.367$; $z_1 = 0.324$ and for the incandescent light $x_2 = 0.4475$; $y_2 = 0.4075$; $z_2 = 0.145$. But

for calculating the mixed colour we require the spacial coordinates $X_1Y_1Z_1$ and $X_2Y_2Z_2$. These are proportional to the triangular coordinates and are obtained from them by multiplying them by certain constants. Now, how must we determine these constants for the mercury light and for the incandescent light? If we remember that in the XYZ system the luminosity is proportional to the Y coordinate (sect. 30), we see at once that we must select the two constant factors in such a manner that $Y_1 : Y_2 = 1 : 2$. This result can be attained, for instance, by selecting the constant 1 for the mercury light, and for the incandescent light $(0.367 : 0.4075) \times 2 = 1.801$.

We find therefore:

$$X_1 = 0.309; \quad Y_1 = 0.367; \quad Z_1 = 0.324,$$
$$X_2 = 0.806; \quad Y_2 = 0.734; \quad Z_2 = 0.261.$$

Addition gives the coordinates of the blended light as follows:

$$X = 1.117; \quad Y = 1.101; \quad Z = 0.585,$$
$$x = 0.3985; \quad y = 0.395; \quad z = 0.2085$$

(see point M in fig. 26).

In case (c) two courses lie open to us. In the first place we can measure the spectral distribution of the light and then work out a calculation as described for case (a). In the second place we can ascertain experimentally in what proportion we must blend any three selected lights in order to match the given light. From these results we can calculate $X\ Y\ Z$ in the same way as this was done in sect. 30, from $B_1B_2B_3$. The means of carrying out these measurements and the advantages and disadvantages of the two methods will be discussed in chapters VIII and IX.

§ 38 *Calculation of the colour coordinates of coloured surfaces*

So far we have only been concerned with calculations on light coming directly from a light source. As a result we have not had to bother much about the total q u a n t i t y of light radiated and the corresponding brightness of coloured surfaces. By bringing the source of light closer, for example, the brightness of a light spot thrown on a screen could be altered at will. The only things that remained unchanged were: the r e l a t i v e spectral distribution of the light penetrating the eye and the colour point in the colour triangle.

Now when we begin to calculate the colour of a coloured surface, or, more accurately, the colour coordinates of the light which the

surface transmits in the direction of the eye under a certain illumination, the same is true, up to a certain point. But if we consider that two differently coloured surfaces, if they are illuminated by the *same* light source, generally display *different* brightnesses, and that the relation of those two brightnesses remains unchanged if we vary the quantity of light radiated by the light source, we see that in this case the brightnesses are certainly a matter of greater concern to us.

We can express the matter thus: when we alter the quantity of light radiated by the light source, *two* quantities now remain constant; firstly the colour points (in the colour triangle) of the light reflected by the coloured surfaces, and secondly the *brightness ratio* of the surfaces.

Let us take for one of these surfaces the so-called "standard white surface", that is a screen covered with a thick layer of magnesium oxide (MgO) applied by holding a plate in the smoke of burning magnesium. This material is the best approach to "ideal white". By a good approximation we mean that it satisfies two requirements, namely:

1. All the light is completely reflected whatever the colour of the incident light may be: there is therefore no absorption.
2. However the surface is illuminated it always assumes the same brightness from whatever direction it is observed (perfect diffuser, absence of gloss or glitter).

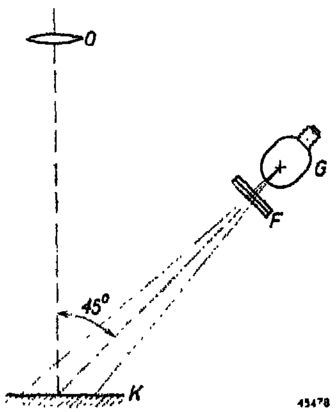


Fig. 27
Illumination and measuring directions when measuring the colour of a surface K. G = incandescent lamp, F = filter (to form illuminant B or C), O = place of measuring instrument or observing eye.

Now if we wish to determine the colour of any given surface numerically, we must first agree on the kind of light with which to illuminate it. In many cases this will be one of the normal lights mentioned in section 23. In other cases we shall be concerned specifically with the colour when illuminated by another light source (for example, mercury light). In the second place we must define the direction from which we shall illuminate it and the direction in which we shall observe it or measure it. This is necessary because, if one of these directions is changed not only may an im-

portant modification in brightness take place (glitter), but in certain cases a change in the character of the colour can also be observed (as with textiles and some kinds of ink). As a rule illumination takes place at an angle of 45° and the observation or measurement from a direction perpendicular to the surface (see *fig. 27*). In certain cases, of course, other directions may be more desirable; they must then be mentioned specifically.

The data from which we start to calculate a colour are of two kinds; in the first place, of course, the relative spectral composition of the light with which we illuminate the surface; in the second place the reflecting properties of the surface. These properties are fixed by giving for each spectral colour the *spectral reflection factor.* By this term we understand the ratio of the brightness which the surface to be studied assumes when illuminated by the spectral colour, to the brightness which the normal white surface would display under the same circumstances.

We represent the spectral reflection factor by the symbol ε_λ.*) As a rule ε_λ is smaller than 1: the normal white surface throws back more light than any other surface. Under special circumstances (highly glossy surfaces under a special choice of illumination and measuring direction) ε_λ can, however, be greater than 1. From these data we can calculate:

a. The colour point of the light radiated by the surface in the direction of the eye.

b. The *reflection factor* ε of the surface, that is the ratio of the brightness assumed by the surface when illuminated by the heterochromatic light to the brightness assumed by the normal white surface under the same circumstances.

A knowledge of ε enables us to determine the constant ratio of the brightness of two differently coloured surfaces when illuminated by the same kind of light.

Now that these conceptions have been established the calculation appears simple. For, if the light falls on the normal w h i t e surface, it will have the same spectral composition after reflection and we would have to determine the coordinates X Y Z as in sect. 37, by multiplying for each λ occuring in the light the power E by the values $\bar{X}$ $\bar{Y}$ and $\bar{Z}$ (table D), and afterwards to add the results for the various wavelengths.

If we now replace the normal white surface by the c o l o u r e d

*) ε (epsilon) is the Greek letter e.

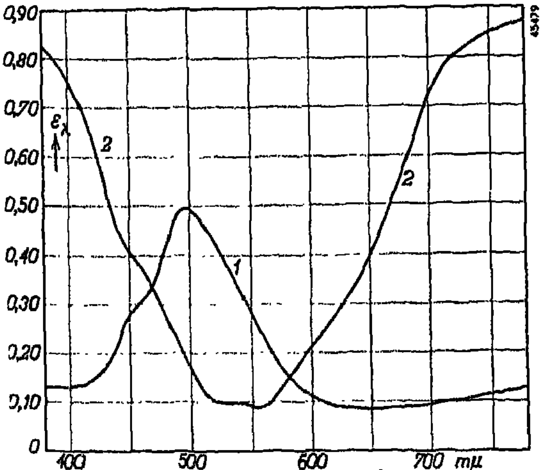


Fig. 28 ⟶ λ

Spectral reflection factor of the coloured papers used in the calculation 1) green paper; 2) purple paper.

surface to be measured, this means that the contribution given by a certain wavelength λ to that addition must also be multiplied by ε_λ. Instead of $E\overline{X}$, $E\overline{Y}$ and $E\overline{Z}$ we must then calculate for each λ the quantities $\varepsilon_\lambda E\overline{X}$, $\varepsilon_\lambda E\overline{Y}$ and $\varepsilon_\lambda E\overline{Z}$. Addition then gives the values X Y Z for the heterochromatic light. The reflection factor ε is calculated by making the same calculation for the normal white surface (here all ε_λ are equal to one) and dividing the Y values of the first calculation by that of the second (these being of course proportional to the brightnesses).

§ 39 *Calculations on coloured surfaces (examples)*

As an illustration of the considerations laid down in sect. 38 we shall calculate the colour of a green slip of paper (from the well-known Ostwald colour atlas, in which it is registered as nc 79) under various illuminations. *Fig. 28* (curve 1) gives the spectral reflection factor for the various wavelengths. If we illuminate this piece of paper with the *mercury light* described in sections 36 and 37 (Hg in fig. 26) the calculation appears as follows:

λ	$E\overline{X}$	$E\overline{Y}$	$E\overline{Z}$	ε_λ	$\varepsilon_\lambda E\overline{X}$	$\varepsilon_\lambda E\overline{Y}$	$\varepsilon_\lambda E\overline{Z}$
578	31·20	31·28	0·06	0·165	5·15	5·16	0·01
546·1	10·74	28·14	0·36	0·301	3·23	8·47	0·11
435·8	7·33	0·39	36·30	0·204	1·495	0·08	7·405
407·8	0·08	0·00	0·40	0·135	0·01	0·00	0·05
404·7	0·27	0·01	1·26	0·134	0·04	0·00	0·17
	according to § 37						

Addition gives for the green colour of the paper: $X = 9.93$;

λ	ε_λ	$\varepsilon_\lambda E\bar{X}$	$\varepsilon_\lambda E\bar{Y}$	$\varepsilon_\lambda E\bar{Z}$
380				0·01
390	0·136	0·01		0·03
400	0·136	0·03		0·14
410	0·136	0·10		0·49
420	0·150	0·42	0·01	2·03
430	0·179	1·25	0·05	6·12
440	0·227	2·26	0·15	11·34
450	0·278	3·09	0·35	16·31
460	0·310	3·41	0·70	19·57
470	0·343	2·87	1·34	18·94
480	0·418	1·93	2·80	16·39
490	0·488	0·84	5·47	12·24
500	0·492	0·14	9·51	8·01
510	0·470	0·29	15·62	4·91
520	0·433	1·99	22·29	2·46
530	0·387	5·07	26·40	1·29
540	0·334	8·33	27·39	0·58
550	0·283	11·40	26·16	0·23
560	0·235	13·97	23·38	0·09
570	0·193	15·77	19 69	0·04
580	0·158	16·57	15·73	0·03
590	0·130	16·25	11·98	0·02
600	0·110	15·07	8·96	0·01
610	0·099	13·54	6·79	0·01
620	0·091	11·17	4·98	
630	0·086	8·33	3·44	
640	0·084	5·94	2·32	
650	0·082	3·85	1·43	
660	0·083	2·35	0·87	
670	0·085	1·33	0·49	
680	0·086	0·75	0·27	
690	0·089	0·39	0·14	
700	0·092	0·21	0·07	
710	0·095	0·11	0·04	
720	0·098	0·06	0·02	
730	0·101	0·03	0·01	
740	0·104	0·02	0·01	
750	0·109	0·01		
		169·2	238·9	121·3

$Y = 13.71$; $Z = 7.74$ or $x = 0.3165$; $y = 0.437$; $z = 0.247$. Addition of the column $E\bar{Y}$ shows 59.82. Therefore we find for the reflection factor:

$$\varepsilon = 13.71 : 59.82 = 0.229$$

If the calculation is repeated for the second kind of mercury light mentioned in section 37 (Hg′ in fig. 26) we find the following:

$$x = 0.286; \quad y = 0.402;$$
$$z = 0.312; \quad \varepsilon = 0.240.$$

Now we shall calculate the colour of the same piece of paper illuminated by incandescent light (Standard illuminant A as light source). The value of $E\bar{X}$, $E\bar{Y}$ and $E\bar{Z}$ we find in table G. These values must now be multiplied by the spectral reflection factor ε_λ, as has been done in the annexed table.

Addition for the green colour gives:

$$X = 169.2 \quad Y = 238.9 \quad Z = 121.3$$
$$x = 0.3195 \quad y = 0.451 \quad z = 0.229$$

Addition of the column $E\bar{Y}$ from table G shows $Y = 1079.0$, so that we find for the reflection factor:

$$\varepsilon = 238.9 : 1079 = 0.2215$$

Finally we shall carry out this same calculation for illumination by *daylight.* The spectral distribution used is given in *fig. 29* and

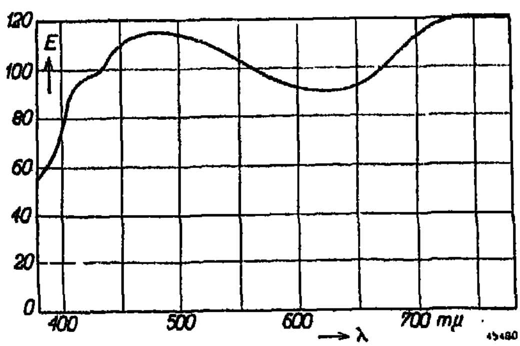


Fig. 29
Relative spectral distribution of the daylight used in the calculation.

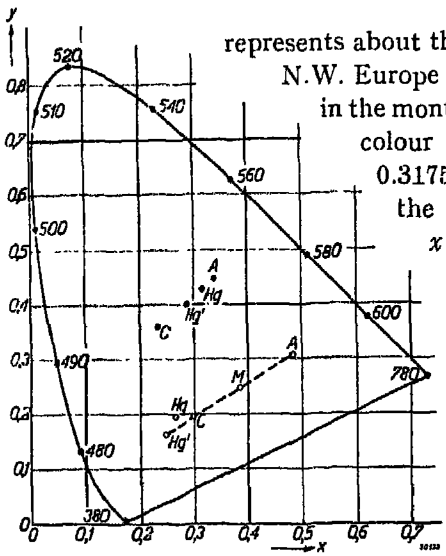


Fig. 30
Calculated colour points of the green paper (·) and of the purplish paper (o) illuminated by various light sources. A = incandescent filament, Hg, Hg' = two kinds of mercury light, C = daylight, M = blended light.

represents about the average colour of daylight in N.W. Europe with a completely overcast sky in the month of December. We find for the colour of this daylight itself: $x = 0.3175$; $y = 0.332$; $z = 0.351$ and for the colour of the green card:

$x = 0.2235$; $y = 0.3575$; $z = 0.265$

In *fig. 30* we have plotted the various calculated green colour points. Hg and Hg' = mercury light, C = daylight, A = incandescent light. It strikes us that the points Hg and A almost coincide. In spite of the enormous difference in the spectral composition of the mercury light and the incandescent light and in spite of the fact that the colour points of the two lights also lie far apart (fig. 26), they both lend almost the same colour to this green card. This example shows how careful one should be in making rough predictions about the colours under differently coloured illuminations. Only careful calculation can show the correct results. Of course the great similarity between Hg and A in fig. 30 is the result of the choice of the coloured paper. For other colours this similarity does not exist at all. If we take, for example, a purple coloured paper (from the same colour atlas, marked nc 42) of which the spectral reflection factors are given in curve 2 of fig. 28, by exactly similar calculations we find the following colour points:

For mercury light (Hg)

$x = 0.260$; $y = 0.201$; $z = 0.539$; $\varepsilon = 0.114$,

for mercury light (Hg')

$x = 0.243$; $y = 0.166$; $z = 0.591$; $\varepsilon = 0.1155$,

for incandescent light

$x = 0.4695$; $y = 0.302$; $z = 0.2285$; $\varepsilon = 0.171$,

for daylight $x = 0.295$; $y = 0.191$; $z = 0.514$; $\varepsilon = 0.271$.

These results are indicated by circles in fig. 30. We see that the accidental similarity between Hg and A no longer exists.

As a final example we shall calculate the colour assumed by the purple paper when illuminated by the blended light we mentioned in sect. 37, *i.e.* mercury light Hg plus incandescent light in the ratio of 1 : 2.

The triangular coordinates of the paper when illuminated successively by mercury light and incandescent light are already at hand. In order to discover the spacial coordinates which after addition give the colour of the paper illuminated by blended light, we must again multiply each set of triangular coordinates by a certain constant factor. If this had been a normal white surface these two constants would have had to be chosen so that the brightnesses of the components — *i.e.* the Y's — stood to each other as 1 : 2. After taking into account the difference in the reflection factor when illuminating the purple paper by mercury light and incandescent light, the ratio of the two brightnesses (and therefore of the Y's) becomes as follows:

$$(1 \times 0.1155) \; : \; (2 \times 0.171) = 1 \; : \; 2.96.$$

This is obtained by multiplying x, y and z for the incandescent light by 1.628. We therefore find for the mercury light contribution $X_1 = 0.243$; $Y_1 = 0.166$; $Z_1 = 0.591$ and for the contribution of the incandescent lamp $X_2 = 0.764$; $Y_2 = 0.4915$; $Z_2 = 0.372$.

If we add these contributions we find for the paper illuminated by blended light:

$$X = 1.007; \quad Y = 0.0575; \quad Z = 0.963 \text{ or}$$
$$x = 0.383; \quad y = 0.250; \quad z = 0.367$$

The colour point (M) is included in fig. 30 and lies as it should on the line AHg'. What figure do we find in this case for the reflection factor?

When we replace the purple paper by the normal white surface we find for Y_1 and Y_2 the values 0.166 : 0.1155 = 1.438 and 0.4915 : 0.171 = 2.875 and therefore $Y = 4.313$. The reflection factor we wished to find is therefore 0.6575 : 4.313 = 0.1525. We obtain this same result in a slightly simpler way if we realize that the blended light consists of one part mercury light to two parts incandescent light; therefore

$$\varepsilon = \tfrac{1}{3}\,\varepsilon_1 + \tfrac{2}{3}\,\varepsilon_2 = 0.0385 + 0.114 = 0.1525$$

For another method of colour calculation see sections 51 and 52.

§ 40 *Relation between the monochromatic and the trichromatic coordinates*

In continuance of sect. 35 we shall now set out to calculate the monochromatic coordinates of a colour from its trichromatic coordinates and vice versa. We can find the colorimetric purity p and the dominant wavelength λ of a colour when its point in the colour triangle is known very easily, though with no very great accuracy, with the aid of *fig. 31*, in which the XYZ triangle lines of constant λ_d (lines through the point $x = z = \frac{1}{3}$) and lines of constant p are drawn. If we inscribe in this figure the colour given by its x and z value, we can at once read off p and λ_d. In *fig. 32* the same diagram has been drawn again, proceeding from the rectangular x, y representation (see fig. 22d).

Supposing we have not got figures such as fig. 31 or 32 at our disposal (or supposing we wish to start from another white point than the E employed there), we can determine the dominant wavelength graphically by drawing a sufficiently large triangle (with the aid of colour table E) in which we indicate the various wavelengths on the curve of the spectral colours [see Hardy[3]].

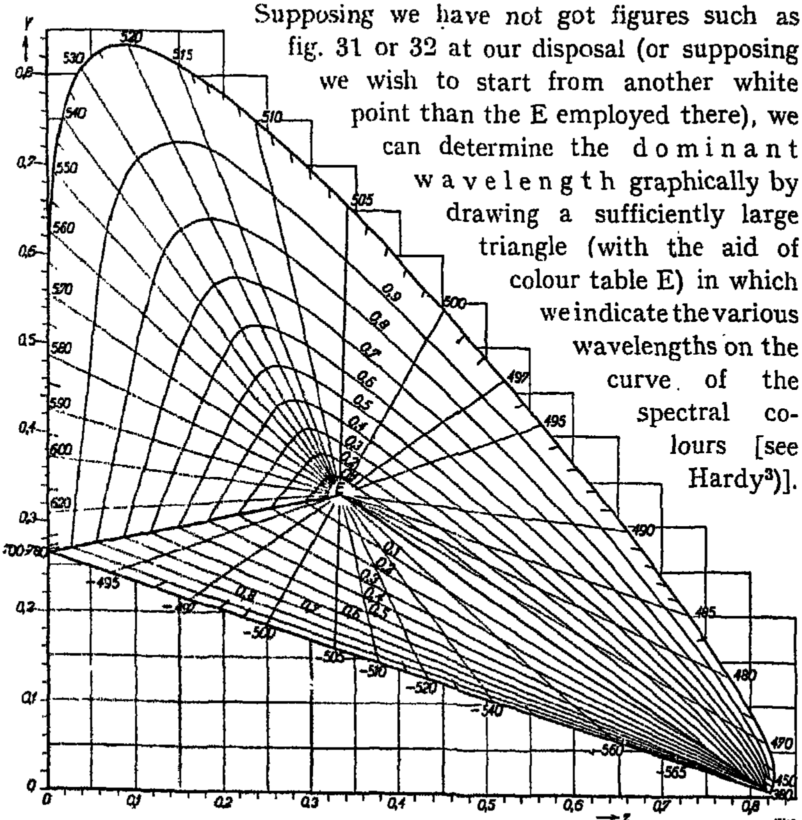


Fig. 31

Chromaticity diagram with lines of constant colorimetric purity p and lines of constant dominant wavelength λ_d (x,z diagram).

Now we join the colour point K to the white point; the connecting line intersects — on extension — the curve of the spectral colours at a certain point. The wavelength inscribed at this point is the dominant wavelength λ_d we wished to find. In *fig. 33* this construction has been given. In this E ($x = y = z = \frac{1}{3}$) has been chosen as white. This simple graphical method, if carried out with care, enables us to determine the value of λ_d for nearly all colours with an accuracy of about 1 mμ (for colours in the neighbourhood of the lines E → 780 and E → 380 (fig. 33) the accuracy is not so great). If we wish to determine λ_d with greater accuracy we must use a different method. We see from fig. 33 that all points on the line EK possess the same λ_d. The dominant wavelength depends therefore solely on the slope of the line EK. This slope we can determine in its turn by the ratio $(y - \frac{1}{3}) : (x - \frac{1}{3})$. For we can easily see that this ratio is also the same for all points of the line EK. But the ratio $(y - \frac{1}{3}) : (x - \frac{1}{3})$ can be easily and accurately calculated from the trichromatic coefficients of K. The only thing, therefore, that we require in order to determine λ_d accurately, is an extensive table giving the relation between the ratio $(y - \frac{1}{3}) : (x - \frac{1}{3})$ and the dominant wavelength λ_d.

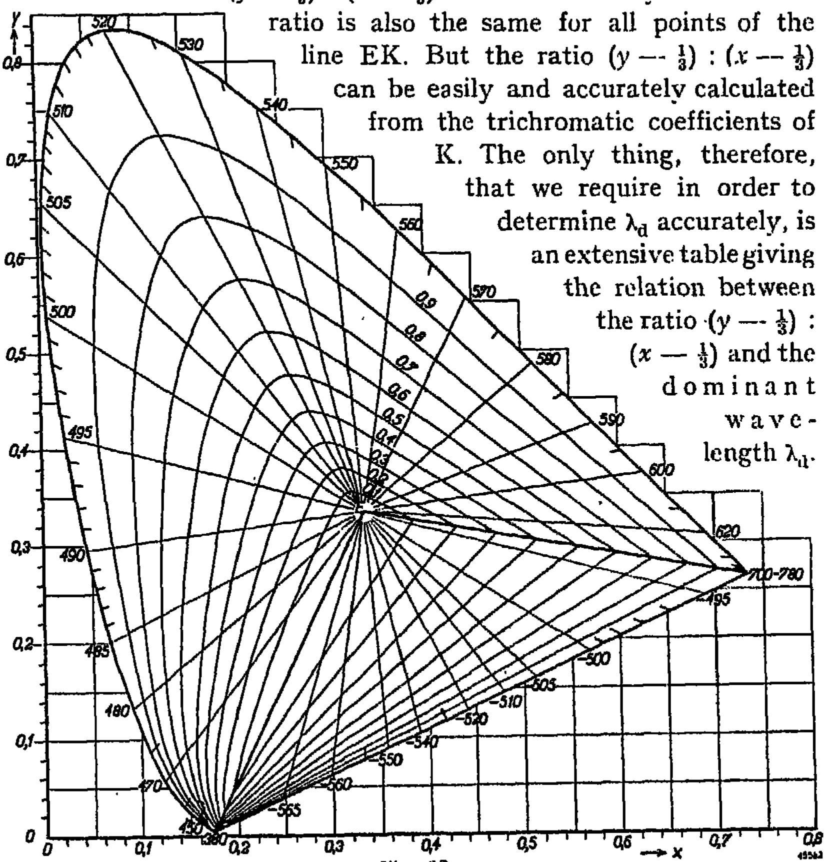


Fig. 32

The same as in fig. 31 but according to fig. 22d (x,y diagram).

In table J we find the required data. In order to avoid the occurrence of very large values of the ratio (this would make the estimation of intermediate values difficult) in all cases in which the ratio would become greater than 1, the inverse value, namely $(x - \frac{1}{3}) : (y - \frac{1}{3})$ has been given. Table J enables us to determine the values of λ_d for almost all colours (provided their trichromatic coordinates have been sufficiently accurately given) with an accuracy of 0.1.

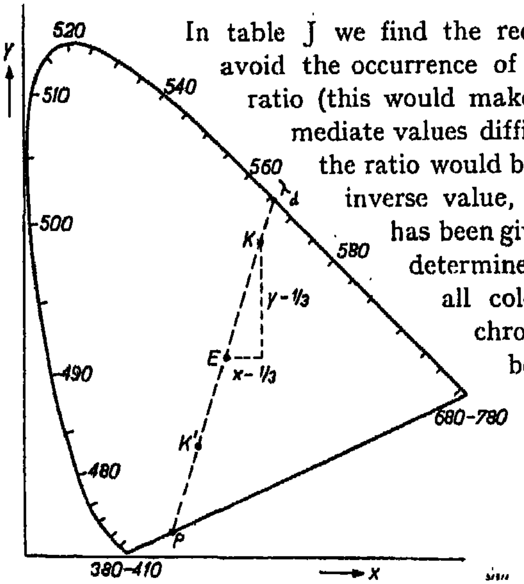


Fig. 33

Graphical determination of the dominant wavelength λ_d of a colour K. λ_d only depends on the slope of EK, *i.e.* on the ratio $(y - \frac{1}{3}) : (x - \frac{1}{3})$

Analogous tables for other white points (A, B, C) are given by Judd [8]).

Finally we wish to draw attention to the fact that to a colour K′ on the extension of KE (fig. 33) the same slope, the same ratio $(y - \frac{1}{3}) : (x - \frac{1}{3})$ and therefore, according to table J, the same λ_d pertains as to colour K. This is in agreement with the previous convention regarding the dominant wavelength of a purple colour (section 22).

We shall now calculate the colorimetric purity p of the colour K. A rough estimate is again possible with the aid of fig. 31 or 32. In order to determine p accurately, we deduce a formula enabling us to determine p from the values x, y and z. If we call the coordinates of the spectral colour possessing the wavelength λ_d, $x_s y_s z_s$ and if we recollect the definition of p — we can match the colour K (brightness B) by mixing λ (brightness pB) and E (brightness $(p - 1)\ B$) — we see (section 31) that the following holds in fig. 33:

$$K\lambda_d : EK = \frac{1-p}{\frac{1}{3}} : \frac{p}{y_s}$$

But geometrically we find

$$K\lambda_d : EK = (y_s - y) : (y - \tfrac{1}{3})$$

From these two equalities it follows:

$$\frac{1-p}{\frac{1}{3}} : \frac{p}{y_s} = (y_s - y) : (y - \tfrac{1}{3})$$

If we solve p from this we find the equation:

$$p = \frac{y_s\,(y - \frac{1}{3})}{y\,(y_s - \frac{1}{3})} \tag{14}$$

In the same manner we can deduce the equivalent relations

$$p = \frac{y_s\,(x - \frac{1}{3})}{y\,(x_s - \frac{1}{3})} \quad (15) \qquad \text{and } p = \frac{y_s\,(z - \frac{1}{3})}{y\,(z_s - \frac{1}{3})} \quad (16)$$

In practice we choose from the equations (14) (15) and (16) that which allows of the most accurate calculation. For instance if y_s is very little different from $\frac{1}{3}$ we do *not* employ equation (14), as in this both the numerator and the denominator are very small and inaccurately known. If we use, exceptionally, another white point instead of E, namely $x_W y_W z_W$, equations (14), (15) and (16) become

$$p = \frac{y_s\,(x - x_W)}{y\,(x_s - x_W)} = \frac{y_s\,(y - y_W)}{y\,(y_s - y_W)} = \frac{y_s\,(z - z_W)}{y\,(z_s - z_W)} \tag{17}$$

If we are dealing with a purple colour (for instance K′ in fig. 33) we must substitute in equations (14)-(17) for $x_s\,y_s\,z_s$ the coordinates of the saturated purple in question (P in fig. 33). We can find the coordinates of P by extending the line EK′ in the colour triangle and reading off the x and y of the point of intersection with the purple line. We can also, starting from the ratio $(y - \frac{1}{3}) : (x - \frac{1}{3})$, calculate the coordinates of P. If we represent this ratio by t then the following equations hold for the x and y of the point P:

$$x = \frac{t - 1.22755}{3\,t - 1.41996};$$

$$y = \frac{0.24677\,t - 0.47332}{3\,t - 1.41996}. \tag{18}$$

In order to deduce equations (18) we need only calculate the point of intersection of EK′ and the purple line. The equations of these two straight lines are:

$$y = -0.07585 + 0.47332\,x;$$

$$y = \tfrac{1}{3}\,(1 - t) + tx.$$

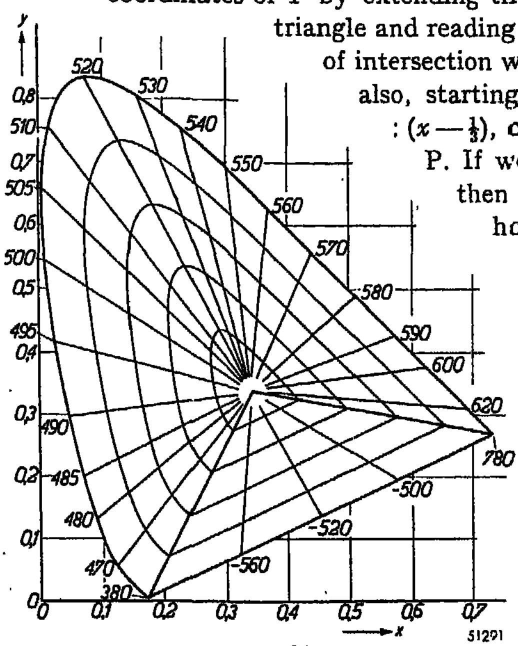


Fig. 34
Chromaticity diagram with lines of constant excitation purity σ and lines of constant dominant wavelenght λ_d.

If, instead of the colorimetric purity (p), we wish to determine the excitation purity (σ) we have at our disposal entirely analogous methods. *Fig. 34* enables us to make a quick but rough determination. From fig. 33 we see that $\sigma = EK : E\lambda_d = (y - \frac{1}{3}) : (y_s - \frac{1}{3})$, so that we obtain the following equations for σ:

$$\sigma = (x - \tfrac{1}{3}) : (x_s - \tfrac{1}{3}) = (y - \tfrac{1}{3}) : (y_s - \tfrac{1}{3}) = (z - \tfrac{1}{3}) : (z_s - \tfrac{1}{3}) \quad (19)$$

If we compare this equation with equations (14)—(16) we see straightaway that the values of p and σ are *only* equal if $y_s = y$, therefore (fig. 33) on the horizontal straight line through the white point. In (19) too, in the case of purple colours, we have to fill in the trichromatic coefficients of the saturated purples (according to equation) (18) for $x_s y_s z_s$.

The inverse problem — the determination of the trichromatic from the monochromatic data — occurs far less frequently. This is a simple mixture problem. For colour K has the same x and y as the mixture of the colour λ (brightness p) and colour E (brightness $1 - p$). We have already solved a similar mixture problem in sect. 37 (case b), so that we need not here go any further into this.

§ 41 *Calculation of p and λ_d (examples)*

We should like to illustrate the considerations of sect. 40 with a few examples. First we shall calculate λ and p for the standard white A (see fig. 26). For this we found (sect. 37):

$$x = 0.4476; \quad y = 0.4075 \text{ and } (y - \tfrac{1}{3}) : (x - \tfrac{1}{3}) = 0.649$$

According to table J we then find that $\lambda_d = 583.45$.

For the determination of p we use formula (16) of section 40:

$$p = \{y_s(x - \tfrac{1}{3})\} : \{y(x_s - \tfrac{1}{3})\} = (0.4640 \times 0.1143) : (0.4075 \times 0.2016) = 0.644$$

We deduce from the values of λ_d and p that standard illuminant A has a decided yellow hue (verging on orange). This is again a case of an apparent contradiction caused by the influence of the environment. In the evening the light in a room illuminated by incandescent light appears to us practically white. If, however, we burn an incandescent lamp (with frosted glass bulb) in clear daylight the colour impression is about what we predicted from the calculated monochromatic coordinates. We see here once more how careful we must be in interpreting such data: they can only give a fairly reliable prediction when the environment in which the object is seen in

practice corresponds roughly in colour with the white used for the calculation of λ_d.

The calculation of λ_d and p for coloured surfaces is carried out in a similar manner. As a first example we shall again take the green piece of paper described in section 39, the spectral reflection factors of which we find in fig. 28 (curve 1).

When illuminated by daylight (see fig. 29) this had the coordinates:

$$x = 0.2235; \quad y = 0.3575; \quad z = 0.419.$$

We therefore find $(y - \frac{1}{3}) : (x - \frac{1}{3}) = 0.2204$ and from table J: $\lambda_d = 494.50$. The value of p we find from (15) of sect. 40:

$$p = \{y_s \ (x - \tfrac{1}{3})\} : \{y(x_s - \tfrac{1}{3})\} = (0.4005 \times 0.1098) : (0.3575 \times 0.3079) = 0.400 \text{ or from (16)}$$

$$p = \{y_s \ (z - \tfrac{1}{3})\} : \{y(z_s - \tfrac{1}{3})\} = (0.4005 \times 0.0857) : (0.3575 \times 0.2408) = 0.400$$

As a second example we shall consider the purple piece of paper of section 39 (fig. 28 curve 2) for which we found, when illuminated by incandescent light:

$$x = 0.4695; \ y = 0.302; \ z = 0.2285$$

We find $(y - \frac{1}{3}) : (x - \frac{1}{3}) = -0.23$ and from table J: $\lambda_d = -494.63$. The minus sign signifies that we have here a purple colour to deal with. From equation (18) we calculate the x and y of the corresponding saturated purple ($t = -0.23$). We find $y_p = 0.249$ and $x_p = 0.700$. Finally we find p from equation (15) of section 40:

$$p = \{y_p \ (x - \tfrac{1}{3})\} : \{y \ (x_p - \tfrac{1}{3})\} = (0.249 \times 0.1362) : (0.302 \times 0.367) = 0.306 \text{ or from (16):}$$

$$p = \{y_p(z - \tfrac{1}{3})\} : \{y(z_p - \tfrac{1}{3})\} = (0.249 \times 0.1048) : (0.302 \times 0.2823) = 0.306$$

The correspondence of the two p values is a check on the correctness of the determination of P (in the previous example of λ_d). The result means that we can match the purple colour by mixing the saturated red purple colour P (brightness 0.306) and the white E (brightness 0.694).

The character of the colour sensation is again different from what one would expect. The colour P lies on the purple line very close to the red, while the colour of the paper makes the impression of lying about as far from the red as from the purple. The reason lies of course again in the fact that we illuminate the surroundings with incandes-

cent lamps whose colour deviates greatly from the standard white E. A better correspondence can therefore be expected if we choose, this time too, standard illuminant A (incandescent lamp) for the white point.

Then we find (for instance by the graphical method) $\lambda_d = -565$ and for the saturated purple P: $y_p = 0.155$ and $x_p = 0.501$.

Finally we find p from (17) of section 40:

$$p = \{y_p(y - 0.4075)\} : \{y(y_p - 0.4075\} = (0.155 \times 0.1055) : (0.302 \times 0.2525) = 0.2145 \text{ or}$$

$$p = \{y_p(z - 0.1449)\} : \{y(z_p - 0.1149)\} = (0.155 \times 0.0836) : (0.302 \times 0.1991) = 0.2145.$$

In fact the point P lies now about half-way between the two extremities of the spectrum.

CHAPTER VII

A few special light sources and colours

§ 42 *Temperature radiation, black body radiation*

One of the most important radiations for practical purposes is *temperature radiation*, by which we understand the radiation of bodies brought to a very high temperature. To this group, of course, belong electric incandescent lamps, but candle-light must also be classified as temperature radiation, the light in this case originating from hot particles of carbon floating in the flame.

The spectral composition of temperature radiation is important to us for the calculation of colours. In order to investigate this we must first take into account another kind of radiation closely connected with temperature radiation, namely *"black body radiation"*. By a black body or total radiator we understand a body that absorbs completely all radiation falling on it whatever the wavelength. It may perhaps appear strange that a "radiator" is defined by a property which at first sight has no connection at all with radiation, namely the power of absorption. Kirchhoff (1860) was the first to see that there is a connection between the two properties. If on the one hand we define as $r\Delta\lambda$*) the power which any given body raised to a high temperature T radiates per cm² in a small wavelength interval λ to $\lambda + \Delta\lambda$ and if, on the other hand, we call a the fraction that the same body, at temperature T, absorbs *of a radiation of wavelength* λ *falling on its surface, then, according* to Kirchhoff's law, the ratio of the "spectral emissivity" r to the absorbing power a is independent of the nature of the body and therefore for all bodies depends in the same way on the quantities λ and T.

Now if we had a good general idea of the way in which this ratio depends on λ and T, we should be able to calculate the spectral radiation directly from measurements of the absorbing power and hence also the spectral distribution of the light radiated at a certain temperature.

*) Δ (pronounce delta), the Greek capital D, is generally used to indicate a very slight increase. $\Delta\lambda$ means therefore a very slight increase of the wavelength λ.

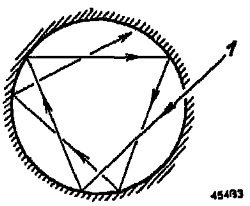


Fig. 35
Realisation of the black body.

The importance of the black body at once becomes obvious here. For in this case $a = 1$ (for each T and λ) so that for the black body the ratio r/a becomes r_b. Therefore if we know the spectral radiative power r_b of the black body as a function of λ and T, by the same token we know the ratio r/a for all temperature radiators.

For the spectral radiative power r_b of black bodies Planck in 1900 gave his famous formula:

$$r_b = E(\lambda,T) = c_1\lambda^{-5} : \left\{ e^{c_2/\lambda T} - 1 \right\} \tag{20}$$

in which the letters have the following significance:

$E(\lambda,T)\Delta\lambda$ = the power radiated per cm^2 in the wavelength interval between λ and $\lambda + \Delta\lambda$ at a temperature T (this power is of course proportional to $\Delta\lambda$).

e = 2.71828.... (a constant that plays an important part in higher mathematics).

c_1, c_2 = the so-called radiation constants; for the *relative* spectral distribution of light only c_2 is of importance. If we express λ in cm, $c_2 = 1.438$.

T = the absolute temperature (measured from the absolute zero at which $T = -273°$ C); the absolute temperature is therefore the temperature in ° C plus 273 (degree Kelvin or ° K).

Planck's formula appears to tally with all experimental data. In order to realize a black body, it is not sufficient to blacken a surface with paint or soot. Such surfaces do not appear to absorb all radiation. We must have recourse to an artifice *(fig. 35)*. We take a hollow opaque body with a very small aperture. Now if we let a beam 1 penetrate the aperture it is reflected several times in the cavity. At each reflection a part will be absorbed. If we make the aperture small enough practically none of the radiation caused by 1 will emerge again finally: all radiation is absorbed, giving us a black body. Then, if we raise the hollow body to a high temperature (for instance by dipping it into a bath of molten platinum) light will radiate from the aperture, the spectral distribution of which is given by formula (20). The standard illuminant mentioned in sect. 17 was realized in the same fashion [see Heller[1]), Dresler[6])]. Since most "temperature radiators" — especially in the range of

visible radiation — behave more or less like black bodies, we can learn a great deal from Planck's formula about the spectral distributions of our light sources and especially of incandescent lamps.

Fig. 36 illustrates formula (20). The variation of E with the wavelength (in the visible range) for temperatures between 1000° and 5000° is reproduced. In order to reproduce the very divergent values in one figure we have chosen a logarithmic scale for E. The corresponding values of E (right) increase by a factor of 10 every time we pass from one horizontal line to the next. We see from this figure how astonishingly fast E increases as we raise the temperature. When we pass from 1000 to 2000° the power rises at 520mμ by a factor of one million! At higher temperatures E increases less quickly with the temperature. This increase of E has two reasons. In the first place the total energy radiated over all wavelengths increases very rapidly with the temperature. From formula (20) we can calculate that this total amount of energy increases with T^4 (Stefan-Boltzmann law) and is therefore 16 times as large for T = 2000° as for T = 1000°. But the chief reason is a different one. Fig. 36 only shows the visible range. If we extended the figure to the right, i.e. in the infra-red region, we should see that all the curves have a *maximum*. For $T = 5000°$ this maximum lies right in the visible range. For lower temperatures it passes more and more to longer wavelengths, thus deeper and deeper into the infra-red. For this maximum we can deduce the following two laws from eq. (1):

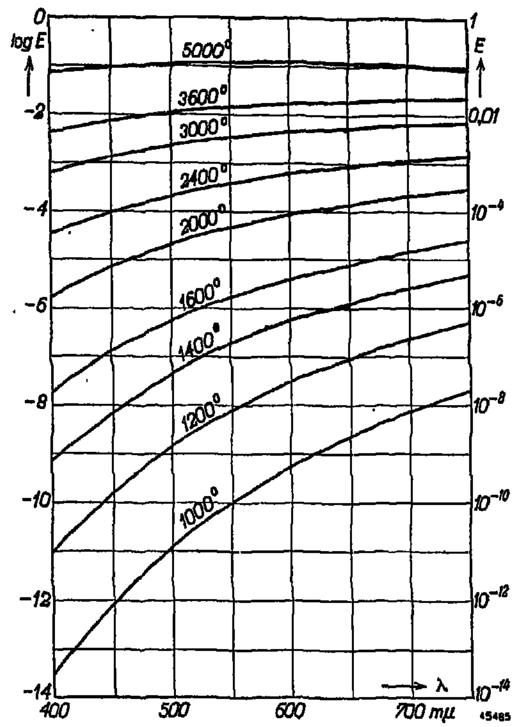


Fig. 36
Radiation of the black body at different wavelengths and temperatures.

1. For the wavelength at which the maximum occurs, the following holds:

$$\lambda T = C \; (C = 2{,}897{,}000)$$

   when we express the wavelengths in mμ.
2. The height of the maximum is proportional to T^5.

It follows from the first law that the maximum only falls in the visible range for temperatures between 3700 and 7600°; when $T = 2000°$ it lies at 1449 mμ, for $T = 1000°$ at 2897 mμ.
The lower the temperature the further is the visible range from the maximum and the greater is the part lost in the infra-red. This is the chief reason for the rapid fall of E with the lowering of T.

§ 43 *Calculation of the colour point of a black body; colour temperature*

In the calculation of colours we are chiefly interested in the *relative* spectral distribution. This can be better seen from *fig. 37*, where E is arbitrarily put equal to 100 at 590 mμ. The shifting of the maximum is again clear: at $T = 2000°$ it lies in the infra-red, at $T = 5000$ in the visible and at $T = 10{,}000°$ in the ultra-violet part of the spectrum.
Below 5000° the red rays predominate in the visible spectrum, above 5000° the blue and violet rays. All this is true for black bodies.
The importance of all this for the study of temperature radiators lies in the fact that for every temperature radiator (for which a is

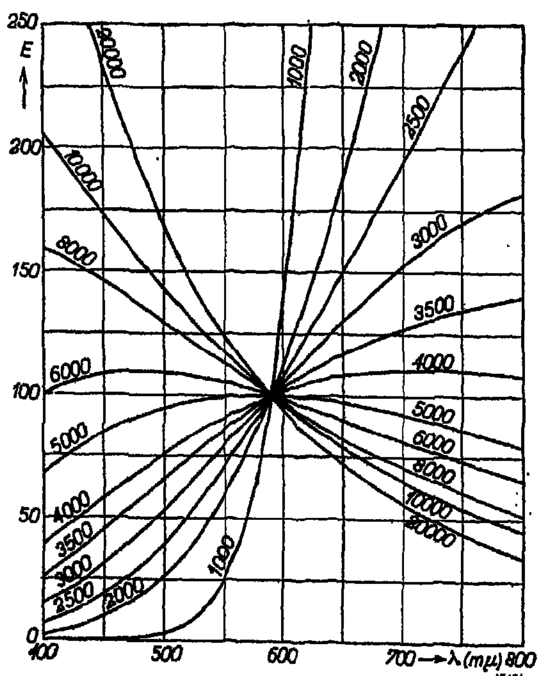


Fig. 37
Relative spectral distribution of black body radiations at temperatures between 1000 and 10,000 °K.

always smaller than 1 and moreover not exactly the same for each λ) a certain temperature value — the *colour temperature* T_c — can be given, which has the following property: in the visible range the relative spectral distribution of the temperature radiator, as a good approximation, is equal to that of a black body at a temperature T_c. This colour temperature T_c is not equal to the true temperature T of the temperature radiator. For instance, T_c of the present incandescent lamps is about 80° higher than T.

Fig. 37 and equation (20) can therefore be used for temperature radiators as well, provided we fill in the values T_c for the temperature. Table 2 lists the colour temperatures of the most common light sources. The *Hefner* candle is the former German light standard, the "new candle" is the present, internationally accepted light standard, *i.e.* a black body at the temperature of melting platinum (see also section 17).

For incandescent lamps the colour temperature is given for the types intended for a voltage of 220 volt. The colour temperature of the corresponding 110 volt lamps lies about 50-100° higher. As a rule the higher of the values given hold for the lamps with the greater power input.

Incandescent gas light has such a peculiar type of radiation that no colour temperature can be attributed to it.

TABLE 2

Light species	T_c
Ordinary candle	1900—1950
Hefner lamp	1880
Kerosene (paraffin oil)	1920—2050
Incandescent gas light	—
New candle	2046
Acetylene	2350—2450
Incandescent electric lamps	
Carbon filament	2100—2200
Tungsten filament:	
Vacuum	2400—2500
Gas-filled (50 W)	2700
ditto (150—500 W)	2800—2900
ditto coiled coil	2700—2750
Cinema and projection	2850—3200
Carbon arc	3700—3800 and higher
Moonlight	4100
Sunlight	5300—5800
Daylight (sun + clear sky)	5800—6500
ditto (overcast sky)	6300—7200
Clear blue sky	14000—50000

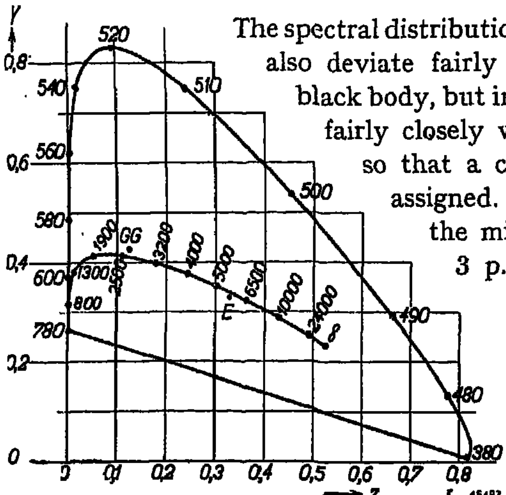


Fig. 38
Black body locus in the y, z chromaticity diagram for temperatures above 800° K.

The spectral distributions of sunlight and daylight also deviate fairly strongly from that of the black body, but in this case the colour agrees fairly closely with that of a black body, so that a colour temperature can be assigned. The figures are valid for the middle of the day (9 a.m. — 3 p.m.) and for fairly normal atmospheric conditions [T a y l o r[1])]. In the early morning and in the evening there are often much lower values (the red of sunrise and sunset). Finally we must point out that with artificial light sources a higher colour temperature generally also indicates a higher true temperature. For sunlight, daylight and moonlight this is not the case. Here the spectral distribution (from which the colour temperature is determined) is influenced by quite different causes, such as light dispersion in the atmosphere, the reflection of the moon's surface, etc. The measurement of high temperatures is also based, in the last resort, on the P l a n c k formula. Hence the uncertainty in the value of c_2 (see section 17) also involves a slight uncertainty in the freezing point of platinum. The value given in table 2 is based on $c_2 = 1.432$. If we take $c_2 = 1.438$ we find 2042° for the freezing point.
We have just seen that we can even speak of a colour temperature for light sources whose colour point coincides with that of a black body, although the spectral distribution differs (daylight, sunlight). It is inadvisable, however, to apply the conception thus extended to spectral distributions which differ greatly from that of a black body. In the literature on the subject various methods have been given for fixing a colour temperature to a light source whose colour point does not quite coincide with that of a black body [D a v i s [1]), J u d d [6, 10])]. These should only be applied to slight deviations too.
The calculation of the colour coordinates of the light from a black body of a given temperature, or of a temperature radiator of given colour temperature takes place as described in section 37 (case a). The figures of table D are multiplied by the E values derived from equation (20), then added and (in order to find the trichromatic

coefficients) the three resulting figures are divided by their sum. In table K the results of such a calculation are given for various colour temperatures and in *fig. 38* they are plotted in the colour triangle. *Fig. 39* gives the same results in the rectangular x,y representation [(see Harding[1,2])].

In table K the dominant wavelengths of the calculated colours are also given. There is no sense in giving values between 5000 and 6500°, since the curve of the black bodies passes the white point E so closely that the colours look practically white. We see from fig. 38 and the λ values that the lower the temperature (below 5000°) the more saturated a yellow colour we obtain. At about 2000° the yellow passes over into a saturated orange and finally becomes red at a colour temperature of about 600°. Above 7000° the light assumes an increasingly pronounced blue colour. At extremely high colour temperatures we approach closer and closer the point indicated in fig. 38 by ∞. These colours can again only be clearly observed when the environment has a colour approximately corresponding to the white point E (see fig. 35). If the whole environment is illuminated by the black body under consideration we shall certainly have to go below about 3000° or above 10,000° to be able to observe a noticeable colour in the light. Between these two rough limits we always call the light white or colourless.

Under different conditions than those mentioned the limits may lie much closer together [see Priest[2]), who places the limits at 4100 and 6200°!]

The light sources mentioned in table 2 we can find for ourselves in fig. 38. Note that in the course of time black bodies of increasing colour temperature have been employed (from candle to "coiled coil" a difference of about 800°). From what has been said in connection with fig. 36 it will surprise no one that during that development the luminous efficiency (the quantity of light divided by the power input to the light source) increased to a great degree. The gas-filled incandescent lamp certainly does not form the end of

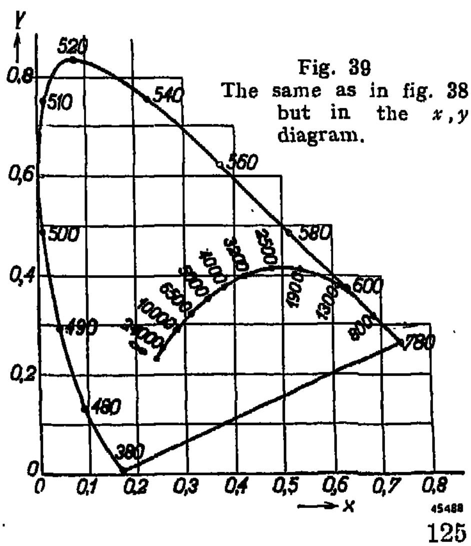


Fig. 39
The same as in fig. 38 but in the x,y diagram.

this rising curve, but from this point no extraordinary improvements are to be expected from the temperature radiators. Here the curve is continued by gaseous discharge lamps, the discussion of which falls outside this chapter.

The colour of the incandescent gaslight is given in fig. 38 by GG. As it does not fall on the line of the black bodies we cannot very well speak of a colour temperature here. The same is true for the white point E.

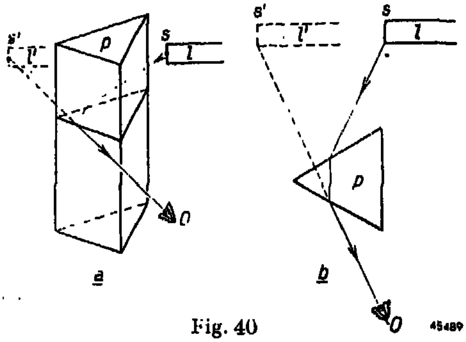


Fig. 40
The eye O looks through a prism (p) at the dividing line (s) between a light strip (l) and its dark surroundings. The observer supposes (l) and (s) to be in the position (l') and (s'). a) in perspective; b) in projection on horizontal plane.

§ 44 *Boundary colours*

We shall now discuss a group of spectral distributions occasionally met with in practice, which moreover show a number of interesting properties causing the corresponding colours to play a great part in the history of colour science (see chapter XI). We refer to the boundary colours which often appear as coloured bands or fringes along the dividing lines of light and shade, as when sunlight plays on cutglass. The coloured fringes we observe in imperfect optical instruments also belong to this same category. We see these colours most plainly when we look through a prism (p) at a dividing line (s) between a light and dark part of the field of vision; in *fig. 40a* (s) represents the vertical dividing line between a white strip (l) and its dark surroundings.

The rays proceeding from (s) change their direction on the way to the eye as a result of *refraction*, at the two glass surfaces, so that the observer thinks he sees the strip (l) with the dividing line (s) in the position (l') and (s'). *Fig. 40b* shows the plan of the experiment.

But we have already seen that the various spectral colours undergo different

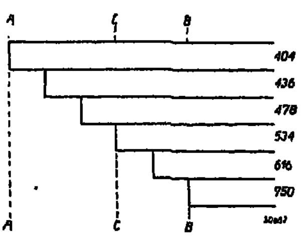


Fig. 41
The position of (l') (see fig. 40) for a number of spectral colours between A and B; colours are seen because only a part of the spectral colours contributes to the forming of an image.

degrees of refraction (section 2). The consequence is that the apparent displacement [from (s) to (s′)] will also be greater for one spectral colour than for another. In *fig. 41* the positions in which we observe (l) and (s) for some spectral colours have been drawn one beneath the other. The wavelengths have been chosen so that each image has been shifted an equal amount relative to the previous one (in a horizontal direction the scale of fig. 41 is about 10 × as large as in fig. 40). What impression does the eye receive on observing all the bands formed by the spectral colours at the same time? To the left of the line A it is dark; no spectral colour penetrates to the left of A. To the right of B it is light and the colour is that which we would observe if we saw band (l) without a prism. In fact all wavelengths are present.

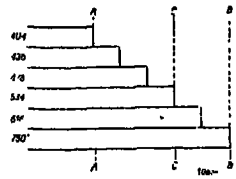


Fig. 42
Schematic representation of boundary colours originating when the light strip of fig. 40 is brought to the left of (l), the position of (s) being unchanged.

Between the lines A and B lies the transition region from light to darkness, in which not only does the brightness steadily decrease from B to A, but we can also observe colour phenomena. All this is the immediate consequence of the fact that at each point between A and B only a part of the spectral colours is represented. Thus the light we receive from C consists entirely of wavelengths shorter than 534 mμ. We may therefore conclude that all "boundary colours" appearing in the transition region between darkness and light can be considered as the original colour emitted by (l), from which all spectral components *above* a certain wavelength have been removed.

But in this way we have only met half the number of possible boundary colours. The other half comes into existence when we exchange the roles of light and darkness in fig. 40. Expressed differently: we leave the position of the dividing line unaltered but now place the white strip (l) to the left of (s). This has the result that in fig. 41 the roles of light and darkness have been interchanged, so that we obtain the scheme of spectral components shown in *fig. 42*.

In this case the light proceeding from C consists exclusively of wavelengths greater than 534 mμ or, more generally, the boundary colours are produced from the original light by the omission of all the spectral components *below* a certain wavelength.

Now when we know the spectral composition of the light emitted by (s) (fig. 40), it is not difficult to determine the colour points of all the boundary colours to be met with. If we choose, for example, the

spectral composition of standard illuminant B (see section 23) to determine the colour, we need only take the distribution curve for illuminant B (see table H) and add the values for the wavelengths smaller than 534 mμ [compare sect. 37 case (a)]. In the same way, for the light we observe in C in the case of fig. 42 we must add up table H from 534 to 780 mμ and calculate the trichromatic coefficients from the three results by dividing by the sum of the three results.

We have divided the distance AB (in figs. 41 and 42) into ten equal parts and determined the colour we observe at the position of the boundary line by the above method. The results are given in table 3. Besides the number of the subdivision the following data are given: the wavelength range present in the particular colour, the x and y of the colour, its reflection factor ε and its dominant wavelength λ_d (this time calculated with respect to the white point B).

TABLE 3

No.	Scheme fig. 41					Scheme fig. 42				
	λ (mμ)	x	y	ε	λ_d	λ (mμ)	x	y	ε	λ_d
0	380—404	0·173	0·005	0.00002	400	404—780	0·351	0·352	1.000	570^5
1	380—419	0·172	0·005	0.00015	416	419—780	0·350	0·355	1.000	571
2	380—436	0·169	0·007	0.0013	427	436—780	0·362	0 377	0.999	571
3	380—456	0·163	0·012	0.0065	442	456—780	0 392	0·431	0.994	571^5
4	380—478	0·150	0·026	0.0226	456	478—780	0·435	0·493	0.977	573
5	380—502	0·133	0·069	0.0705	470	502—780	0·470	0·511	0.930	575
6	380—534	0·126	0·200	0.252	483	534—780	0·526	0·472	0.748	582
7	380—571	0·186	0·336	0.599	491	571—780	0 622	0·378	0.401	599
8	380—616	0·300	0·360	0.902	495	616—780	0·710	0·290	0.098	632
9	380—674	0 347	0·352	0.997	495	674—780	0·734	0·266	0.0035	689
10	380—750	0 348	0352	1.000	495^5	750—780	0·725	0·265	0.00002	700

The wavelength ranges in table 3 are to some extent dependent on the nature of the prism glass. In calculating table 3 normal crown glass was used with the following refractive indices:

λ	768.2	656.3	589.3	486.2	434.1
n	1.5116	1.5146	1.5171	1.5233	1.5282

In *fig. 43* the boundary colours appearing under these circumstances have been plotted in the usual manner. The curve VB contains the boundary colours of the scheme of fig. 41 (left of table 3), the curve BR those of fig. 42 (right of table 3). The figures of the subdivisions are also shown in the figure. With the assistance of fig. 43 and the calculated values of ε and λ_d we can get a good idea of the boundary colours usually to be met. Passing from V to B we first get violet

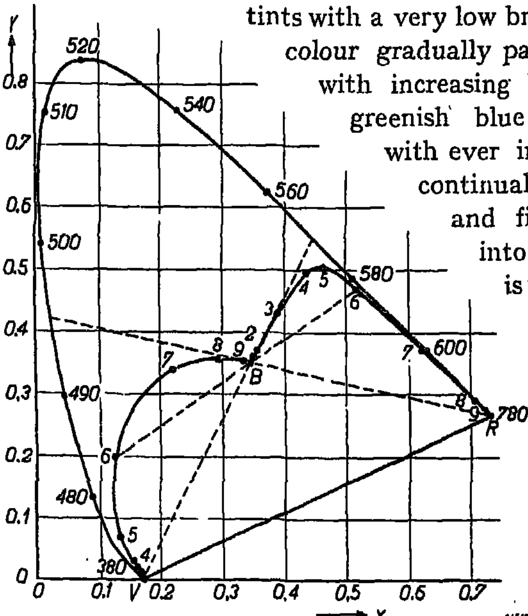


Fig. 43

Boundary colours in the chromaticity diagram (illuminant B). The colours on VB denoted by 4-9 correspond to the left half of table 3 (see fig. 41), those on BR to the right half (see fig. 42).

tints with a very low brightness (1– 3); then the colour gradually passes over into pure blue with increasing brightness (4—6); then greenish blue colours appear (7—9) with ever increasing brightness but continually smaller saturation, and finally the colour passes into white. This, therefore, is the colour sequence which appears in the coloured fringe AB of fig. 41. If we pass from B to R (fig. 43) the sequence is as follows: greenish yellow with increasing saturation and slowly decreasing brightness (1–4); saturated yellow (5–6); then, with gradually decreasing brightness, orange (7) orange red (8) and red (9). These colours appear in the colour fringe AB of fig. 42.

Because the brightnesses in the blue-violet band (B—V) decrease much faster than in the yellow-red band (B—R), on looking through the prism we get the impression that the first band is narrower than the second (in the blue band the brightness has already sunk to 7 % of the maximum in the middle, subdivision 5, while in the other band an equally low brightness only occurs after point 8).

As fig. 42 is produced from fig. 41 "by interchanging light and darkness", a colour in the fringe of fig. 42 contains exactly those spectral colours which are missing in the corresponding colour of the fringe of fig. 41 (table 3 also shows this at once). If we mix these corresponding colours additively they will yield the complete illuminant B precisely. In other words, the two corresponding colours are *complementary*. We can ascertain this fact in fig. 43. The two colours marked 6 lie on a line with B.

According to what we have just said, a colour situated on the branch BV in fig. 43 but close to B lies on the straight line connecting B with a colour lying close to R. Hence it follows that the curve VB touches the straight line BR at B. Similarly the line BV is the

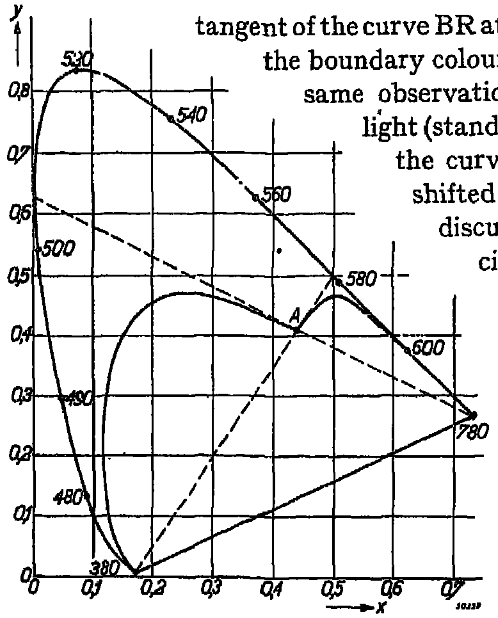


Fig. 44
Boundary colours with incandescent electric light (illuminant A).

tangent of the curve BR at B. Finally *fig. 44* illustrates the boundary colours we obtain if we make the same observations by incandescent lamp-light (standard illuminant A). Although the curves have now of course been shifted, the considerations we have discussed above apply, in principle, unaltered. The apparent difference in breadth of the two bands has not been discussed here. Boundary colours were extensively studied and described by G o e t h e. We shall return to this in section 70.

§ 45 *Combinations of boundary colours*

The colours described above are met at every dividing line between light and darkness that runs parallel to the three parallel ribs of the prism (see fig. 40a). Such dividing lines [such as (s)] are displaced by the prism in a horizontal direction, and as this displacement is different for the various spectral colours, coloured edges appear. If, however, we look at a dividing line running perpendicular to the three parallel prism ribs (horizontal lines in fig. 40a) we see no colours. Here each point of the dividing line is displaced in the direction of that line. It is true that here, too, one spectral colour is more displaced than another, but all coloured bands produced by the separate points of the dividing line overlap each other on that same line in such a way that the result is always colourless. It will have struck the reader in the description of the boundary colours that *green* is missing. This is confirmed in table 3. For the green colours in the spectrum we gave at the time the approximate limits 495 and 566, and none of the λ_d values in table 3 lies within these limits. This absence of green is one of the many differences which distinguish the series of boundary colours from that of the spectral colours. Other differences are that the boundary colours are mixed colours and some of them very desaturated. We shall see presently how green may be produced by a combination of boundary

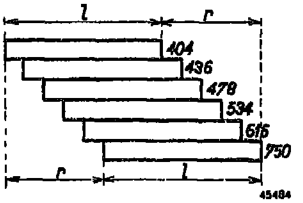


Fig. 45
Boundary colours of a light strip limited on both sides (case $l > r$). On the left the colours of fig. 41 appear, on the right those of fig. 42.

colours and how, with the aid of such combinations, we shall even be able to produce a complete transition from boundary colour to spectral colour. In figs. 40-41 we only studied the colour phenomena appearing on the left border of the strip (l). If we now take into account the coloured edge appearing on the right border of (l) we shall see that in certain cases more complicated colours can appear. This depends on whether the length of the strip (call it l) is greater or less than the distance over which the boundary colours proceeding from one dividing line (AB in figs. 41 and 42) extend. This latter distance we call r. The simplest case occurs when l is greater than r. If we then draw the displacement of (l) for the various wavelengths (analogous to figs. 41 and 42) we get the scheme of *fig. 45*. To the left we see the phenomenon of fig. 41 over a distance r (violet and blue boundary colours). Then follows the unchanged spectrum over a distance $l - r$. Finally we see to the right over a distance r the phenomenon of fig. 42 (yellow, orange and red boundary colours). We therefore see that the original white strip has shrunk to a length $l-r$ and has a coloured edge on both sides. Now if $l = r$ the white part entirely disappears and the two coloured edges join each other.

Now what shall we see if l is smaller than r? It is clear that the two coloured edges then overlap each other in one way or another. As an example in *fig. 46* the positions of the strip are drawn for various wavelengths on the supposition that $l = 0.6\ r$. From the figure we read the following: from A to C some of the boundary colours of fig. 41 are found. At C the light contains all wavelengths smaller than 534 mμ. But from C to E a new phenomenon arises. Wavelengths are now absent from the light both on the long-wave and the short-wave side of the spectrum. Thus the light we observe at D contains only the wavelengths between 436 and 616 mμ. Finally between E and F the only spectral colours missing are on the short-wave side of the spectrum, so that we see part of the boundary colours of fig. 42. Outside the range AF no light is to be observed. From the diagram of fig. 46 the trichromatic

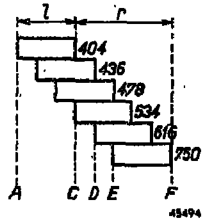


Fig. 46
Boundary colours of a light strip limited on both sides (case $l < r$). Between C and E combinations of boundary colours occur.

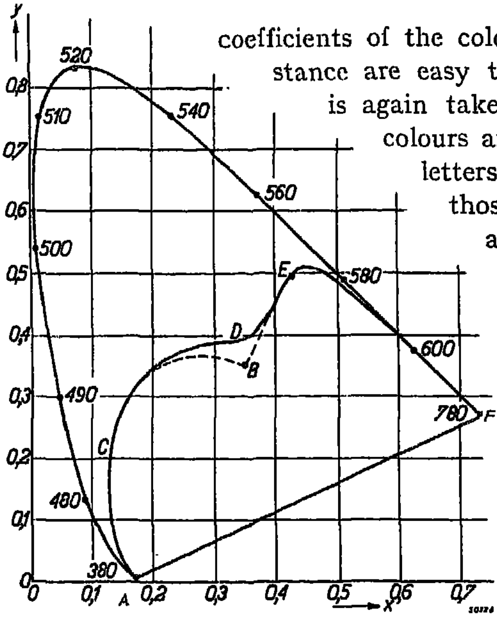


Fig. 47

The colours of fig. 46 in the chromaticity diagram. AC and EF represent boundary colours; between C and E combinations of boundary colours.

coefficients of the colours appearing in this instance are easy to calculate. Illuminant B is again taken as a light source. The colours are plotted in *fig. 47*. The letters ACDEF correspond with those of fig. 46. From A to C and from E to F the colours lie on the curve previously found (fig. 43) but the portion CDE diverges from the old curve, as we have a new type of colour to deal with in this space.

It appears from fig. 46 that these colours are produced by the combination of two systems of boundary colours overlapping each other. We should be inclined to say, by mixing the colours which would have been produced on that spot by both systems. But here we must be cautious. We can indeed call it a mixture but it is in this case not an additive mixture but a special kind of subtractive mixture (see section 6). The best known example of a subtractive mixture is that in which a pencil of rays is sent successively through two filters, one behind the other. One filter reduces a part of the spectral components and afterwards the second filter reduces a part of the remainder. If we study the colour produced at D (fig. 46) we see that this is quite an analogous process. If only the left border of (1) were present only the components with wavelengths greater than 616 mμ would have disappeared from the light, but because of the existence of the right border wavelengths smaller than 436 mμ were removed from the remaining light.

In figs. 45-47 we studied a white strip on a dark background. If, on the other hand, we observe a dark strip on a white background we must apparently exchange light and darkness in figs. 45 and 46 too. It is obvious that in the case in which l (length of the dark strip) is greater than r we see the dark strip shrunk to a length $l - r$ with a system of boundary colours on both sides (this time the red-yellow edge to

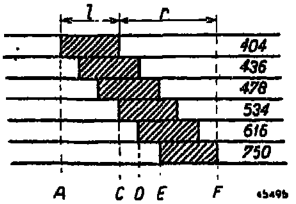


Fig. 48
Boundary colours of a dark strip, limited on both sides, on a bright background (case $l < r$). Between C and F new combinations of boundary colours appear.

the left). If $l = r$ then the dark piece between has vanished and the coloured edges meet. If l is smaller than r the circumstances sketched in *fig. 48* arise (in which the shaded part represents darkness), which was produced from fig. 46 by again interchanging light and darkness. The figure indicates that, from A to C and from E to F, there are again parts of the boundary colour systems of figs. 42 and 41, but from C to E there is again a new phenomenon. Here a wavelength range is missing from the middle of the spectum; at D, for example, the wavelengths between 436 and 616 mμ are missing. On calculating the trichromatic coefficients of the colours displayed we obtain *fig. 49*; from B to C on the curve found previously (fig. 43), from C via D to E the new types of colour, and thence from E to F back to the white point.

If we gradually decrease the length of l in figs. 46 and 47 we shall diverge increasingly from the original curve of fig. 43. The central section will rise continuously and for very small l approach closer and closer to the curve of the spectral colours. This latter was to be expected, for if we look through a prism at a very narrow strip of light we see an ordinary spectrum.

If, in the case of the dark strip on a light background, we decrease the value of l we first pass along the original curve of fig. 43 (for $l = r$); for smaller l we traverse an ever narrowing loop (illustrated in fig. 49), until finally, for very small l, the loop shrinks to the white point. This latter phenomenon was also to be expected. A very

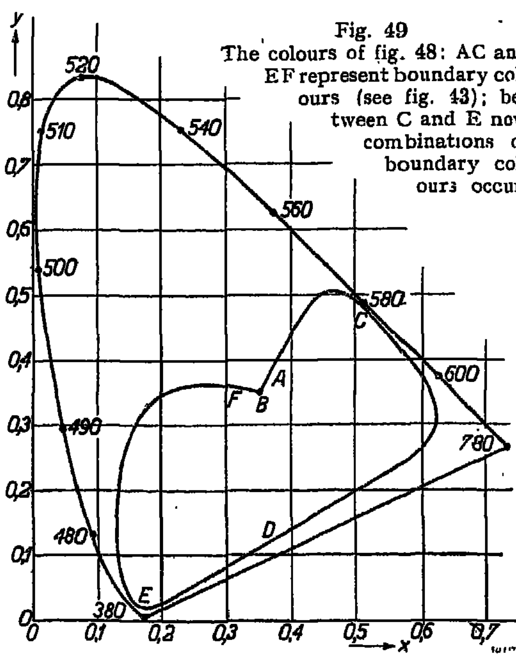


Fig. 49
The colours of fig. 48: AC and EF represent boundary colours (see fig. 43); between C and E new combinations of boundary colours occur.

narrow dark strip cannot exert any influence worth mentioning, either as regards colour or brightness, on the large, bright surrounding field. *Fig. 50* shows the form of these curves for various values of the ratio $l:r$. The upper part of the figure contains the colours seen when observing a white strip against a dark background, the lower those of the opposite case.

We also see at once from fig. 50 that when combining boundary colours there can be no question of additive mixing. With additive mixing the saturated green would never be produced, as there are no two boundary colours that lie on one line with saturated green (green lying between them). We see therefore the confirmation of our earlier observation that the simple mixing laws we deduced for additive mixing do not hold for subtractive mixing.

§ 46 *"Ideal" and "optimal" colours*

All the colours plotted in fig. 50 were produced by omitting certain parts from the continuous spectrum of a particular light source (here illuminant B). We can clearly also produce colours possessing similar spectral distributions by causing the original light to be reflected by a coloured surface with a suitable spectral reflection curve (see sect. 38), such as one of the shapes sketched in *fig. 51*. Reflection curves of type *c* only reflect the spectral colours below a certain wavelength and therefore produce the colours we knew in fig. 43 (branch VB) as boundary colours. In the same way type *d* corresponds with the boundary colours on the branch BR of fig. 43 (yellow-orange-red).

A surface with a reflection curve of type *a* reflects only the spectral colours between two

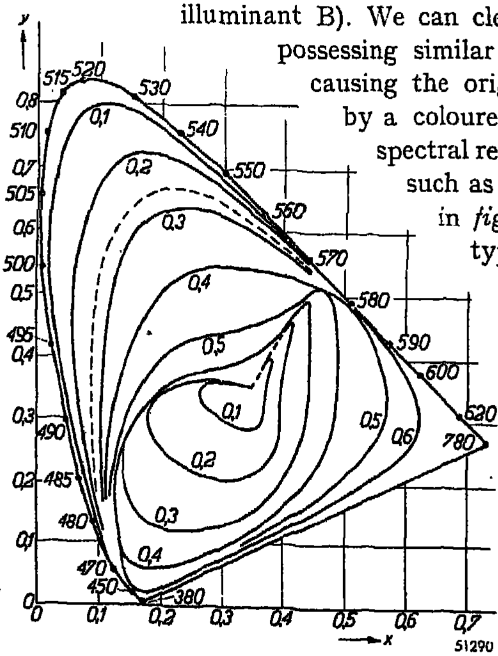


Fig. 50

The complete system of boundary colours and combinations of boundary colours occurring when l/r in figs. 46 and 48 is made to take on all values between 0 and 1

particular wavelengths, and therefore produces the same colours which appeared in the combination of boundary colours when we looked at a white strip on a dark background. These colours lie above the two curves of boundary colours in fig. 50. In the same way type *b* corresponds with the colours which lie below the two curves of boundary colours (combinations of boundary colours seen when looking at a dark strip on a white background).

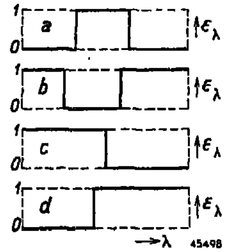


Fig. 51
The four types of spectral reflection curves belonging to the so-called ideal colours.

It will be clear that the excision of certain parts of the spectrum illustrated in fig. 51 can also be brought about by passing the light through a filter the transmission curve of which possesses one of the shapes *a-d*. We say of coloured surfaces with reflection curves of these types and of the corresponding filters that they display "ideal" colours,

> although, properly speaking, there is nothing "ideal" about these colours. It is the reflection or transmission curves and therefore the spectral distributions that are "ideal".

In practice "ideal colours" do not of course occur. The reflection curves will never jump so abruptly from the value 0 to the value 1. But there are, as a matter of fact, coloured objects whose reflection curves approach those of the ideal colours fairly closely.

We shall now discuss an important and remarkable property of the ideal colours. We start from a certain kind of light with a continuous spectrum (e. g. illuminant B) and illuminate any selected coloured surface with it. The reflected light will have a certain spectral composition corresponding to a certain point k in the colour triangle. The same point k can also be attained, according to fig. 50, by the combination of two boundary colours, and, as we have seen above, a coloured surface can also be indicated having one of the reflection curves of fig. 51, which, illuminated by the chosen light source, will produce point k. Hence we see that quite different reflection curves can correspond with the same point in the colour triangle. This is not surprising if we remember the fact that the collection of spectral distributions is much more extensive than that of the colour points.

If we consider all reflection curves leading to point k in combination with the selected light source, we see that the colours produced can only differ in brightness. We can now ask the question, which of all these reflection curves produces the colour with the greatest

Fig. 52
A reflection factor which in a certain wavelength range is neither 0 nor 1 cannot give an optimal colour.

brightness. We say of the surface with this reflection curve that it possesses an "optimal" colour. Now the answer to the question is surprisingly simple. The greatest brightness is attained when the reflection curve is one of the types of fig. 51. In a word, "ideal" colours are at the same time "optimal" colours.

This remarkable property holds for any selected light source with a continuous spectrum and for any point k. The existence of this general property was first proved completely by Schrödinger[1]). We shall give a short account of the principles on which this proof is based. If we wish to show that an "optimal" colour can only be attained by a reflection curve of one of the types of fig. 51 we must prove two things:

1. A reflection curve of the type of *fig. 52* — where in a certain wavelength range, ε_λ is neither 1 nor 0 — can produce no optimal colour. This is shown by selecting in this wavelength range three smaller intervals (shown black in the figure). The chromaticities represented by each of the intervals could be mixed in such a proportion that the chromaticity of k was matched (Grassmann's first law, see section 24). Hence it follows that by altering ε_λ in these three ranges in the right manner we can add a certain quantity of the colour species k. We can, therefore, by altering the curve of fig. 52, increase the brightness of the colour species k. This curve is therefore not yet optimal. An optimal colour can therefore contain no ranges in which ε_λ is neither 1 nor 0. But then the reflection curve must be of the type of *fig. 53*.
2. A reflection curve of the type of fig. 53 cannot produce an optimal colour as long as there are at least 3 wavelengths at which ε_λ jumps from 0 to 1 or from 1 to 0.

   This again follows from Grassmann's first law. The chromaticity k can be reproduced by a suitable mixture of the wavelengths λ_1, λ_2, λ_3. Hence, by shifting the places where ε_λ "jumps" a little in the right direction (for instance, towards the positions given by dotted lines) we can add a certain quantity of the chromaticity k. We can, therefore, by changing the curve of fig. 53, increase the brightness of the chromaticity k. Therefore this curve does not produce an optimal colour either.

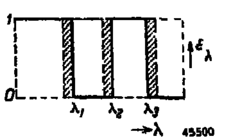


Fig. 53
A reflection curve with ε_λ either 0 or 1, but jumping at more than two places cannot give an optimal colour.

We therefore come to the conclusion that the

reflection curve producing an optimal colour must everywhere have the value $\varepsilon_\lambda = 1$ or $\varepsilon_\lambda = 0$, and that ε_λ must "jump" at no more than two places. But only the 4 types given in fig. 51 comply with these demands.

The proof of the thesis on the ideal and optimal colours is only given sketchily above, chiefly with the aim of giving an idea of a method of proof that can be applied to various colour problems. If we examine the proof more closely we see that in some cases it does not apply, for instance, if the colour points of the three small wavelength ranges in figs. 52 or 53 lie on one straight line and k does not lie on that straight line. Such cases cause the statements "each ideal colour is an optimal colour" and "each optimal colour is an ideal colour" to be not without exceptions. It happens that a colour point k can be attained by several ideal colours which are not always all of them optimal colours. And sometimes by way of exception a non-ideal colour may happen to be an optimal colour. When, however, the thesis is formulated thus: "among the one or more optimal colours having their colour point in k at least one ideal colour occurs", it holds quite generally.

Further it is tacitly understood in the proof that ε_λ values greater than 1 do not occur. The proof therefore only holds for diffusely reflecting surfaces.

§ 47 *Applications of the theory of ideal and optimal colours.*

When considering the ideal and optimal colours in section 46 we were chiefly occupied with the reflecting properties of coloured surfaces. It is, however, clear that quite analogous considerations are valid for the transmitting qualities of filters. In both cases part of the light is lost and the only difference between the filter and the coloured surface lies in the different directions in which the remaining part of the light is propagated. We can therefore draw important conclusions from what has been said above both for filters and for coloured surfaces. In manufacturing and studying pigments one of the problems most frequently occurring is how to make a pigment which, when a certain light source is used, produces a given chromaticity with as great a brightness as possible, therefore with the greatest possible ε. And what are the limits above which ε cannot possibly be varied? The answer to the first question is clear. Try to find a pigment whose reflection curve resembles as much as possible one of the curves given in fig. 51. As in practice such curves can never be completely realised it is of importance to find out how far an existing pigment is removed from the theoretical ideal. Here the second question asserts itself, to which the answer is: Find the accessory ideal colour for the given chromaticities and calculate the reflection factor. This factor forms the theoretical maximum that can never be exceeded with actual pigments. In *fig. 54* these

maximum values are given for illumination with incandescent light (illuminant A) and for illumination by sunlight (illuminant B) in *fig. 55*.

Analogous figures for the standard light sources C and the German E are to be found in Mac Adam [1]) and Richter [6]).

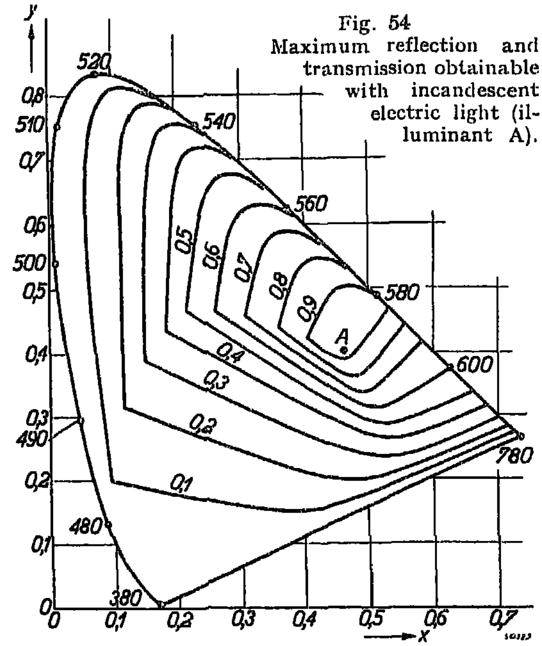
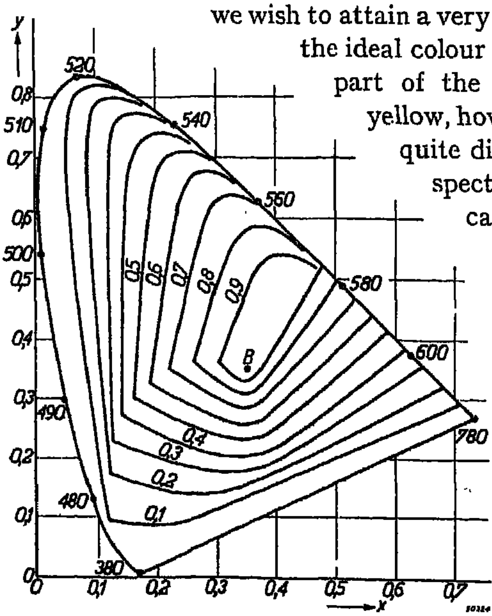


Fig. 54
Maximum reflection and transmission obtainable with incandescent electric light (illuminant A).

In both figures we see that very desaturated colours can appear with high brightnesses. In producing these colours we can allow almost all spectral components to play a part. But as more saturated colours are produced fewer spectral colours are involved and so we can only attain a smaller ε, therefore a lower brightness. If, for instance, we wish to attain a very saturated green colour, then the ideal colour only contains a very small part of the spectrum. For a saturated yellow, however, the circumstances are quite different. As the curve of the spectral colours runs straight we can still use an appreciable portion of the spectrum. Saturated yellow colours can be obtained with higher brightnesses than other equally saturated colours.



We can represent in colour space the whole set of colours that may

Fig. 55
Maximum reflection and transmission obtainable with sunlight (illuminant B).

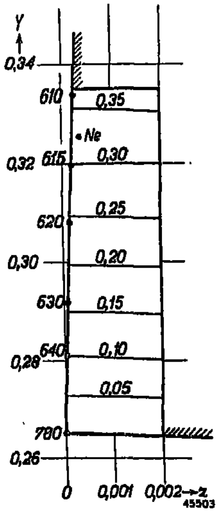


Fig. 56
The region of aviation red in the y, z chromaticity diagram with the maximum obtainable transmission of coloured filters (illuminant A). Ne = colour point of neon light.

appear when illuminating all imaginable coloured, diffusely reflecting surfaces with the same kind of light and the same illuminating strength. As any colour can be produced but in this case a certain brightness may not be exceeded, this set is called a "colour solid", having as boundary surfaces the cone of the spectral colours (extended by the purple surface) and a curved plane containing the colour points of all optimal colours.

We repeatedly come across such colour solids, especially in German publications [see Luther[1]], Nyberg[1]), Richter[6]), Frieser[1]), Neugebauer[1]) etc.] As a rule the XYZ space is not used but a linear transformation of this which, in many cases, displays the unpleasant attribute that the curve of the spectral colours cuts the straight line at infinity at one or two points in the accessory colour surface [Richter[6]), Frieser[1])]. The consequence of this is that the surface representing ideal colours consists of two parts which only meet at infinity!

When studying coloured *filters* — especially traffic signals for land, sea and air — the analogous problem is of still greater importance. Consider again a particular light source (usually an incandescent lamp) and a particular required chromaticity. For economic reasons it is very desirable that the coloured filter used allows the transmission of as great a portion of the light as possible. In other words its *transmission factor* should be as high as possible. Fig. 54 shows immediately how great the theoretically attainable maximum is for each chromaticity. *Figs. 56* and *57* will serve as illustrations. Here two ranges of the colour triangle are shown within which — by international agreement — the colour points of the lights used for aerial traffic as red and as yellow signals must lie.

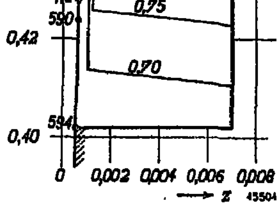


Fig. 57
The region of aviation yellow in the y, z diagram, with maximum transmission of filters (illuminant A) Na = colour point of yellow sodium lines (the sodium lamp proper is outside the diagram z = 0.026).

The red range is limited by the conditions $y<0.335$, $z<0.002$, the yellow range by $0.402 < y < 0.460$, $z < 0.007$. In both ranges the maximum transmission factor is given for the various colours (see also fig. 110).

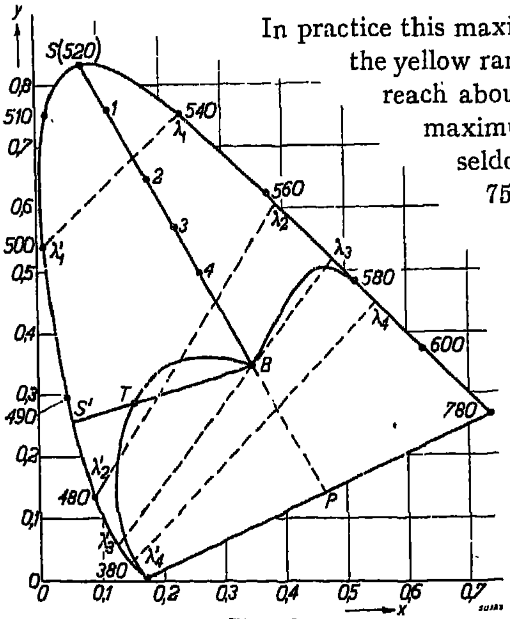


Fig. 58
The situation of the characteristic colour on the line SB. An ideal colour having its point on SB may be matched by a mixture of illuminant B (brightness B_B) and the spectral colour S (brightness B_S). The quantity B_S has its maximum value in 3. Thereore 3 is the C-colour. The limiting wavelengths λ_3 and λ'_3 are complementary.

In practice this maximum is never reached. In the yellow range the best existing filters reach about 85 % of the theoretical maximum, while the red filters seldom attain to more than 75 % of the theoretical transmission. In the case of green filters conditions are much less favourable. A filter having 30 % of the theoretical value must be counted as a good one. This difference in the practical attainment of ideal filters for various colours can be explained. The red and yellow filters require an approximate imitation of the type of transmission curve sketched in fig. 51d; for the green filters, however, we have to try to reproduce fig. 51a. Now it is much easier to find pigments which cut off all wavelengths below a certain limit fairly sharply than pigments showing a more or less sharply defined transmission range on both sides.

Finally we wish to draw attention to the fact that for signalling purposes discharge lamps are coming more and more into use beside incandescent lamps with filters. Thus, for instance, the colour point of the neon-lamp (Ne) lies in the red range and that of the sodium lines (Na) in the yellow range. Both neon-lamps and sodium-lamps have the advantage of having a much greater luminous efficiency than the combination of incandescent lamp and filter.

§ 48 *The C-colours. "Characteristic colours" or full colours*

Among the set of "ideal" colours we find another group of examples distinguished from the rest by their special properties. In the German language these colours are designated by the word "Vollfarben" (full colours). The best formulation seems to me to be "the most

colourful examples among the ideal colours", while perhaps the term "characteristic" colours might serve. We shall call them C-colours. When moving (*fig. 58*) from the white point (in this case standard illuminant B) along a straight line to a point S on the curve of the spectral colours we find that the colorimetric purity steadily increases. If we now examine the ideal colours whose colour points lie on BS in surroundings illuminated by the light B, we may wonder which example of this series of colours makes the most colourful impression, or, in other words, which example has the most pronounced characteristic property of the whole series, that is, the "greenest". The characteristic colour does not lie close to B (the colours are not pure enough there) nor close to S (even the ideal colours have too low a brightness there). The colour sought must therefore lie somewhere about halfway between W and S and form a compromise between the contradictory demands of high brightness and high purity.

The question as to the position of the characteristic colours is actually far removed from the sphere of the problems hitherto dealt with. Such questions lie entirely in the realm of colour sensations. The oft-mentioned limit to the study of colour has clearly been overstepped here in the direction of psychology. Still an obvious effort has been made to bring this psychological problem into relationship with the other properties of colours which can be fixed exactly on the basis of the known mixture laws.

If we consider that — at a certain illumination strength — each of these ideal colours can be matched by a mixture of a brightness B of the "colourless" standard white B and a brightness B_S of the "pure" colour S it is more or less self-evident to suppose that the characteristic colour requires the greatest brightness B_S for its reproduction. It has not yet been sufficiently investigated whether this supposition is quite correct. Probably the greatest divergences will be in the red. Here the colour with the maximum B_S gives the impression of being by no means the most colourful. When the colorimetric purity is increased the red colour becomes more pronounced. If, however, we accept this supposition the problem is reduced to a relatively simple mixing problem. We see at once that B_S in the neighbourhood of point B approaches zero. But also when we approach S, B_S decreases as the total brightness of the ideal colour decreases up to S, and B_S is always smaller than that total brightness. Between B and S there is therefore a point where B_S is maximum. This point represents the characteristic colour of the selected series. In order to determine the position of that point we shall find out

what wavelength range the ideal colour must reflect to include the various points of BS.

If we first choose a point 1 close to S it is apparent that a reflection curve of the type of fig. 51a is required with a fairly narrow range λ_1 — λ_1' in which ε_λ equals 1.

If we pass to a point 2 further removed from S this can be attained by extending the wavelength range where ε_λ equals 1 to both sides. By choosing the ratio of these two extensions suitably we shall succeed in remaining on the line SB. Now what happens to B_S during these extensions?

When passing from 1 to 2 we add the spectral ranges $\lambda_1 \rightarrow \lambda_2$ and $\lambda'_1 \rightarrow \lambda_2'$. These two ranges produce, additively mixed, a colour whose point lies within the quadrangle $\lambda_1\lambda_2\lambda'_2\lambda'_1$, and as a matter of fact on the straight line SB (since 1 and 2 lie on it). The colour we added can therefore be replaced by a mixture of the colours B and S. It appears from this that a positive amount has been added both to B_S and B_B. We have therefore not yet reached the characteristic colour. This increase of B_S remains until the line $\lambda_2\lambda'_2$ has arrived at the position $\lambda_3\lambda'_3$, therefore until the two limiting wavelengths λ_3 and λ'_3 have become complementary. If we continue a colour is added composed of the ranges $\lambda_4 \rightarrow \lambda_3$ and $\lambda'_3 \rightarrow \lambda_4'$. But the colour point of these added colours lies on the extension of SB and can therefore only be replaced by a mixture of the colour B and a negative amount of colour S (see section 21). The consequence is that B_S now decreases. In the language of colour equations:

For point 3: B_3 (colour 3) $\leftrightarrow$ B_S (colour S) + B_B (colour B).

For the added colour 3':

B_3' (colour 3') + B_S' (colour S) $\leftrightarrow$ B'_B (colour B).

For point 4 therefore (addition of the two equations):

B_4 (colour 4) + B_S' (colour S) $\leftrightarrow$ B_S (colour S) + $(B + B_B')$ (col. B) or B_4 (colour 4) $\leftrightarrow$ $(B_S - B_S')$ (colour S) + $(B_B + B_B')$ (colour B).

We see therefore that B_S has indeed decreased to $B_S - B_S'$.

Hence it appears that B_S in point 3 has passed through a maximum. The characteristic colour has therefore a reflection curve according to 51a with two complementary wavelengths which limit the range where ε_λ equals 1. Table 4 gives as

TABLE 4

p	$\lambda \rightarrow \lambda'$	B_B	B_S
1.000	520—520	0.000	0.000
0.944	500—540.10	0.013	0.227
0.792	480—561.64	0.101	0.384
0.666	470—574.86	0.207	0.412
0.511	460—589.40	0.364	0.380
0.352	450—604.03	0.544	0.296
0.207	440—619.36	0.724	0.189
0.091	430—637.28	0.876	0.088
0.000	380—780	1.000	0.000

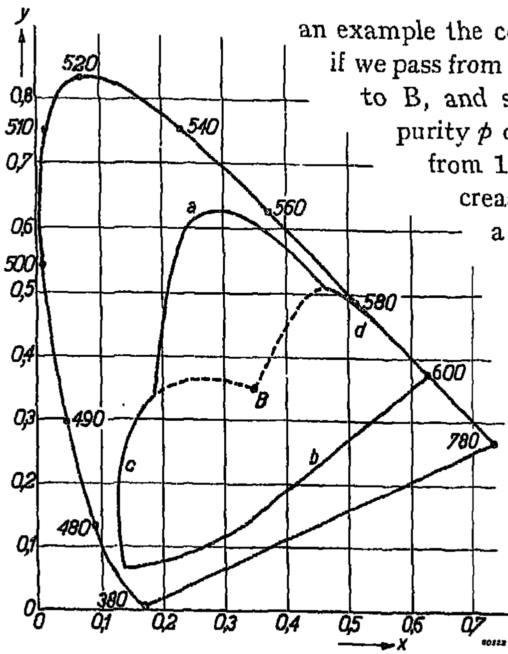


Fig. 59
The complete collection of C-colours obtained with illuminant B. The types *a* to *d* correspond to the four types of reflection curves of fig. 51.

an example the course of λ, λ', B_B and B_S if we pass from the point S (here $\lambda = 520$) to B, and so allow the colorimetric purity p of the ideal colour to drop from 1 to 0; B_B continually increases but B_S passes through a maximum near the characteristic colour.

It may happen, however, that before λ_3 and λ'_3 become complementary one of the two has reached the end of the spectrum. This takes place when S'B intersects the curve of the boundary colours (fig. 58, see also fig. 43) before B_S has reached its maximum value. Until this intersection is reached B_S steadily increases; after passing it B_S decreases again; so in this case the characteristic colour is reached just at the line of the boundary colours and the characteristic colour has a reflection curve of the type of figs. 51c or 51d. Finally, if we find out where the characteristic colour lies on a line such as PB, we obtain, by quite analogous reasoning, the result that this characteristic colour has a reflection curve like fig. 51b with two complementary wavelengths. We can therefore sum up the results as follows: all characteristic colours are either ideal colours with two complementary wavelengths or boundary colours with a limiting wavelength with no complementary.

All characteristic colours have been inscribed in *fig. 59* starting from standard illuminant B. The letters *a-d* correspond to the types of reflection curves of fig. 51. The dotted line gives the boundary colours. The characteristic colours form the starting point of the colour system drawn up by O s t w a l d (see section 75). An important part of their properties was pointed out by L u t h e r[1]).

CHAPTER VIII

Objective colorimetry

§ 49 *Introduction. Objective and subjective colour measurement*

In the course of time countless methods and instruments have been invented for measuring the colour both of light proceeding directly from a light source and of light reflected by a coloured object. We cannot hope to give any complete survey of this subject. The restrictions which we shall impose upon ourselves will chiefly be of two kinds. In the first place we shall restrict ourselves to methods enabling us to define colours with reasonable accuracy in terms of the trichromatic system (XYZ) which was explained in the previous chapters. In the second place we shall chiefly occupy ourselves with the principles on which the various methods are based, while the technical difficulties and the ingenious ways of overcoming them in the construction of the instruments will only be mentioned very briefly. The reader who wishes to study the latter more closely is referred to the appropriate publications on the subject (if, for instance, he wishes to construct such an instrument himself).
As with photometry (sect. 17), so also for colorimetry we can divide the instruments and methods into two main groups: objective or physical colorimetry, which is chiefly based on purely physical measurements, and subjective or visual colorimetry in which ample use is made of the properties of the eye of the observer for colour vision. There are cases when it is doubtful whether a particular method belongs to the first or to the second group. To distinguish more sharply we shall agree to consider as belonging to objective colorimetry those methods in which the eye has work to do, the results of which are not dependent on the individual properties of the observer's eye. By this we mean in the first place the reading of electrical measuring instruments and the position of movable parts of instruments, and also the adjustment to equal brightness of the two halves of the field of vision, provided the two halves radiate light of exactly the same spectral compositon (for instance, spectral colours of the same wavelength). For in this last instance, too, the results

of various observers will agree even if there are persons among them with strongly divergent relative sensitivity curves, or who are abnormal in colour vision, or even quite colour blind.

Of the two groups mentioned that of subjective colorimetry is of course the older. The whole standardized C.I.E. system (cf. sect. 30) is as a matter of fact due to the results of subjective measurements, and only after the introduction of the C.I.E. system could serious efforts be made to invent objective methods yielding results which were in agreement with the system.

The present state of affairs is such that in practice we must insist that both subjective and objective colorimetry furnish results in agreement with the C.I.E. system.

It is of course self-evident that in principle it is easier for objective colorimetry to comply with this requirement than for subjective. In the latter there is always the question of the observer's eye, whose properties never correspond exactly with the C.I.E. standards laid down in the tables. Moreover Kohlrausch [2]) has shown that these properties undergo periodical changes with the seasons. The same periodical phenomenon was shown by Dresler [3]) for the relative sensitivity curve (measured by the flicker method). These fluctuations are not very great, it is true, but they can be clearly observed. For precision measurements the subjective methods have the further disadvantage that the eye sets a limit to the accuracy of adjustment by not being able to observe colour differences smaller than a certain amount (limen). On the other hand, in using objective methods it is in principle possible — and already attainable in practice — to differentiate between colours lying considerably closer. As against the disadvantages of subjective colorimetry there are also disadvantages in objective measurements. In the first place very delicate and costly instruments are required in order to realize the high accuracy possible in principle, and these can only be worked by a staff skilled in the problems of experimental physics. Moreover, many objective measuring methods have the disadvantage of requiring a great deal of arithmetic.

It will therefore depend on all kinds of practical circumstances whether we prefer a method from one group or the other in any particular case. It will, for instance, depend on the accuracy required whether a few measurements or a whole collection are necessary, or whether skilled assistants are available or whether we have access to certain instruments etc. etc. It is therefore to be expected that both groups will remain in existence side by side, and that all efforts will

be made to eliminate by further improvements the disadvantages of both methods.

As the disadvantages of subjective colorimetry are, as a rule, more a matter of principle and therefore more difficult to overcome objective colorimetry (as also objective photometry) has, in my opinion, the best chances for the future.

For this reason we shall deal with this group first.

§ 50 *Spectrophotometry*

The most obvious objective method is the determination of the spectral energy distribution of the light proceeding from the light source or from the coloured surface, followed by the calculation of the trichromatic coordinates with the aid of the tables given at the end of this book.

The measurement of the spectral distribution of energy can be divided into the following steps:

1. The formation of the spectrum of the light rays to be measured, that is the analysis of the light into its spectral components.
2. The measurement of the intensities of these components.

For the first purpose a prism is almost always used. We allow the pencil of light to fall on a small aperture S in an opaque screen *(fig. 60a)*. If we make all light rays proceeding from a point S parallel by means of a lens L_1 and then focus them in a point S′ by means of a second lens L_2, we produce in S′ an image of S. Let us now place the prism P (*fig. 60b* between L_1 and L_2). As the parallel pencil of rays falling on P still consists of parallel rays even after leaving the prism, the lens L_2 will again produce an image S′ at the same distance from the aperture S. Compared with fig. 60a this image has shifted. But now the amount of this displacement is different for the various spectral components, so that instead of an image S′ we have a continuous series of images lying next to each other in the plane AB, each containing its own spectral colour. From the *spectrum* thus produced we can now separate a special wavelength (or rather a very narrow spectral band) by placing in the plane AB an opaque

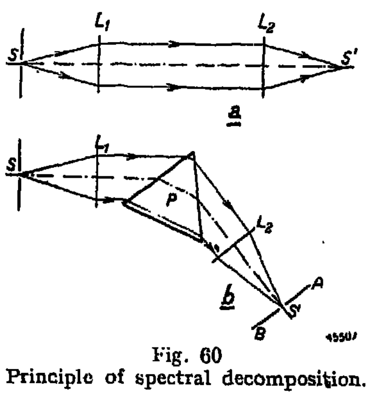


Fig. 60
Principle of spectral decomposition.

screen with a second aperture S′ in the plane AB, which only transmits the spectral light falling on this point of the spectrum. By passing S′ across the plane AB, or by turning P and L_2 together through a small angle we can bring each wavelength in succession to the position of S′. We can also transpose the spectrum in the plane AB by shifting the aperture S to the side.

In the above we have only dealt with the principle on which spectral analysis is based. We cannot discuss here further improvements and refinements such as the use of more complicated prisms or the use of several prisms.

Now the intensity ratios of the various wavelengths must be measured.

If we wish to compare the ratio of the power possessed by each of the spectral colours directly, various experimental complications crop up. Thus, amongst other things, we must take into account the fact that in displacing S′ the isolated spectral band $\lambda \rightarrow \lambda + \Delta\lambda$ does not remain the same size: the value of $\Delta\lambda$ alters. Further we must be absolutely sure of the sensitivity of the measuring instrument employed to the various wavelengths. All such complications are avoided by comparing the spectrum to be investigated, wavelength by wavelength, with a spectrum of known spectral composition. If it is intended to determine the spectral reflection curve of a coloured *surface* the spectrum proceeding from the coloured surface must, as a matter of course, be compared with the spectrum proceeding from a normal white surface (see section 38) illuminated by the same kind of light and at the same illumination intensity. If we measure the ratio of the intensities with which one wavelength occurs in these two spectra, we at once obtain the spectral reflection factor for that wavelength.

If the spectral composition of the light of a light source is to be measured, that of a known light source is chosen as a comparison spectrum, preferably a black body, for which it is only necessary to know the colour temperature in order to be able to calculate the spectral distribution (sect. 42). One of the standard illuminants A, B or C may also be used.

Working in this way one deals exclusively with comparisons between two quantities of light of the same spectral species. This comparison can be effected by various measuring instruments. The only requirement is that the instrument should possess sufficient sensitivity to each spectral colour.

The following instruments are among those sensitive to light: photo-

electric cells and thermo-elements (producing an electric current which can serve as a measure for light sensitivity), photographic plates (the density is a measure of the quantity of light) and the eye (adjustment to equal brightness).
When using the photo-electric cell we measure the two lights in succession. If the photo-electric cell is linear (that is to say if the electric current or voltage is proportional to the amount of light) then the light ratio is simply the ratio of the two currents or voltages. If, however, we have a photo-electric cell or another detector which is not linear (photographic plate, eye, etc.) we can measurably weaken one of the quantities of light to be compared until the instrument indicates that the two are equal. The reduction factor required now gives the desired ratio (of course the same is possible with a linear detector).
When using the photographic plate we first photograph the unknown spectrum and then the spectrum of the known light source; the latter in a series of various intensities. We can now trace for each wavelength the points at which the photographs of the known and unknown spectra correspond (*i.e.* produce the same density on the plate). If the eye is used to make the comparison, we must arrange the instrument so that we see the two quantities of light *simultaneously* as two adjacent parts of the field of vision. We now adjust to equal brightness by a measured reduction of one of the two halves. We can effect this reduction and the required form of the field of vision in countless ways. For these reference must be made to the numerous publications [among others, McNicholas[1]), Ley[1]), Ornstein[2]), Gibson[2, 3]), Sewig[1])].

In conclusion another remarkable instrument must be mentioned [Hardy[5]), Michaelson[1]), Gibson[4])], which was developed in America and which is capable of measuring and recording the spectral reflection curve of a coloured surface quite automatically. In this instrument the light of a particular spectral colour — isolated from the spectrum of an incandescent lamp — is allowed to fall alternately on the coloured surface V and on the normal white surface N in quick succession. A photo-electric cell (F) receives these alternating lights. If F receives equal quantities of light from N and V then the photo-electric cell produces a constant electric current. If, however, the two quantities are unequal then an alternating current is produced which, after amplification, works a small electric motor (M). This motor works a mechanism that reduces the quantity of light falling on F until V and N again produce the same current. Then F stops its alternating current supply and the motor stops. The size of the revolution performed by M is a measure of the required weakening and therefore also a measure of the spectral reflection factor of V. A second motor M′ then moves the aperture separating the wavelength from the spectrum slowly through the spectrum. At each

position of the aperture the motor M ensures equality of the quantities of light produced by V and N.
M′ also turns a registration drum uniformly. The pen that draws a curve on this revolving drum is driven by motor M and gives in each position of the drum the spectral reflection factor of the transmitted wavelength. In this way the spectral reflection curve is registered quite automatically in a few minutes.
For the manner of illumination of the surface to be measured by this instrument see Knipe[2]).

§ 51 *Calculation of colour points from the measured spectral distribution*

In order to determine the trichromatic coordinates of lights or surface colours from the measured. spectral distribution of a light source or the spectral reflection curve, we must carry out the calculations which have already been extensively discussed in chapter VI. We can therefore restrict ourselves here to giving a few methods of shortening these often long and tedious calculations.

In the first place we shall consider the method of selected ordinates. Earlier in the book the following intructions were given for the calculation of the colour point of the light radiated by a light source with a continuous spectrum: divide the spectrum into a large number of equal wavelength intervals (for instance, 10 mμ), determine for each interval the products $E_\lambda \overline{X}_\lambda$, $E_\lambda \overline{Y}_\lambda$, $E_\lambda \overline{Z}_\lambda$ (E_λ from the spectral distribution, $\overline{X}_\lambda$, $\overline{Y}_\lambda$, $\overline{Z}_\lambda$ according to table D) and find the XYZ of the light by summation:

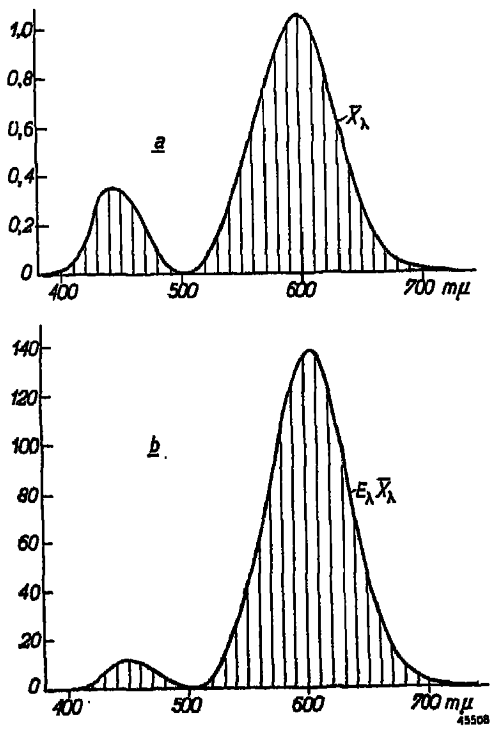


Fig. 61
One of the distribution curves for the equal energy spectrum ($\overline{X}_\lambda$ in fig. 61a) and for illuminant A ($E_\lambda \overline{X}_\lambda$ in fig. 61b). The spectrum is divided into intervals of 10 mμ.

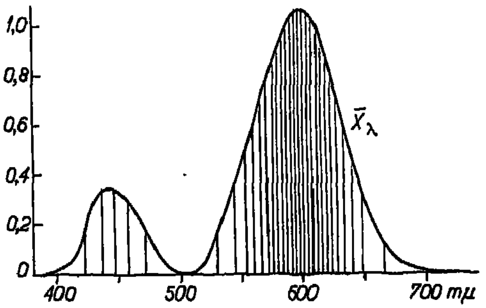


Fig. 62
The same curve as in fig. 61a but now the area under the curve is devided into equal strips (principle of the method of selected wavelengths).

$$X = \Sigma E_\lambda \bar{X}_\lambda,$$
$$Y = \Sigma E_\lambda \bar{Y}_\lambda, \quad (21)$$
$$Z = \Sigma E_\lambda \bar{Z}_\lambda.$$

If the intervals are taken in 10 mμ sizes, there are 100 multiplications to be made besides the addition. We shall now show how to save the trouble of all those multiplications by the method of selected ordinates. In *fig. 61a* the quantities $\bar{X}_\lambda$ (table D) and in *fig. 61b* the quantities $E_\lambda \bar{X}_\lambda$ are plotted for the various wavelengths (for E in this example the spectral distribution E_A of standard illuminant A has been selected, see table F). In both figures the spectrum has been divided into intervals of 10 mμ. In fig. 61a the areas of the blocks are proportional to $\bar{X}_\lambda$, in fig. 61b to $E_\lambda \bar{X}_\lambda$. The sum $\Sigma E_\lambda \bar{X}_\lambda$ (formula 21) is therefore proportional to the whole area lying in fig. 61a below the curve, while the area below the curve in fig. 61b is equal to $\Sigma \bar{X}_\lambda$; this is therefore the quantity we require to determine the X coordinate of the equal energy spectrum, which we shall indicate by X_E.

Now in these figures the blocks have been selected of equal widths, but it is of course also possible to calculate the area below the curve by summing blocks of unequal width. We now take *(fig. 62)* for the curve $\bar{X}_\lambda$ a division of the spectrum into 30 blocks all with *the same area* (viz. $\frac{1}{30} X_E$); as $\bar{X}_\lambda$ increases the blocks therefore become narrower.

If we apply the same division to fig. 61b the blocks no longer have the same area, but their area is proportional to E_λ. The total area below the curve $E_\lambda \bar{X}_\lambda$ is therefore:

$$X = \Sigma E_\lambda \cdot \tfrac{1}{30} X_E = \tfrac{1}{30} X_E \cdot \Sigma_{x} E_\lambda, \qquad (22)$$

since X_E is the same constant for all blocks and can therefore be placed before the Σ sign.

We have now substituted equation (22) for (21).

In order to calculate X by equation (22) we must indeed make an addition, but the hundred multiplications have disappeared.

The expression $\Sigma_{x} E_\lambda$ means that we must find the value E_λ from the

spectral distribution for the centre of each block from fig. 62 and add these 30 values. The centres of these blocks are called *selected ordinates* for the curve $\overline{X}_\lambda$; we find their values in table L, where they are indicated by λ_x.

In exactly the same way we find:

$$Y = \tfrac{1}{30}\, Y_E \sum_Y E_\lambda \text{ and } Z = \tfrac{1}{30}\, Z_E \sum_Z E_\lambda.$$

Here we must add the values of E_λ for the wavelengths λ_y and λ_z, which are obtained in the same manner by dividing the area below the curves $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$ into 30 blocks of equal areas. These selected ordinates are also given in table L. Finally we find:

$$\begin{aligned} X : Y : Z &= \tfrac{1}{30}\, X_E \sum_X E_\lambda : \tfrac{1}{30}\, Y_E \sum_Y E_\lambda : \tfrac{1}{30}\, Z_E \sum_Z E\lambda \\ &= \sum_X E\lambda : \sum_Y E\lambda : \sum_Z E\lambda \end{aligned} \tag{23}$$

for $X_E = Y_E = Z_E$.

We can apply an entirely analogous method for coloured surfaces whose reflection factors have been measured and whose colour points when illuminated by one of the standard illuminants A, B or C are required.

If we choose illuminant A as an example we find by the old method:

$$X = \Sigma \varepsilon_\lambda E_\lambda \overline{X}_\lambda; \quad Y = \Sigma \varepsilon_\lambda E_\lambda \overline{Y}_\lambda; \quad Z = \Sigma \varepsilon_\lambda E_\lambda \overline{Z}_\lambda,$$

in which the summations take place over a number of equally wide blocks. We now draw the curves $E_A\overline{X}\lambda$, $E_A\overline{Y}\lambda$ and $E_A\overline{Z}\lambda$ (see table G) and divide the spectrum into 30 wavelength intervals in such a manner that the area situated below the three curves is divided every time into 30 parts of equal area. The area of one part is therefore $\frac{1}{30}\, X_A$, $\frac{1}{30}\, Y_A$ and $\frac{1}{30}\, Z_A$ respectively.

If we apply this block division to the curves

$$\varepsilon_\lambda E_A \overline{X}_\lambda \qquad \varepsilon_\lambda E_A \overline{Y}_\lambda \qquad \varepsilon_\lambda E_A \overline{Z}_\lambda$$

the area of these blocks becomes proportional to

$$\varepsilon_\lambda \cdot \tfrac{1}{30}\, X_A, \qquad \varepsilon_\lambda \cdot \tfrac{1}{30}\, Y_A \text{ and } \varepsilon_\lambda \cdot \tfrac{1}{30}\, Z_A.$$

The total area below these curves is therefore:

$$\begin{aligned} X &= \Sigma\, \varepsilon_\lambda \cdot \tfrac{1}{30}\, X_A = \tfrac{1}{30}\, X_A \Sigma \varepsilon_\lambda, \\ Y &= \tfrac{1}{30}\, Y_A \Sigma \varepsilon_\lambda,\ Z = \tfrac{1}{30}\, Z_A \Sigma \varepsilon_\lambda \end{aligned} \tag{24}$$

$$\text{or: } X : Y : Z = X_A \sum_X \varepsilon_\lambda : Y_A \sum_Z \varepsilon_\lambda : Z_A \sum_Y \varepsilon_\lambda, \tag{25}$$

so that we have once more found equations in which the 100 multiplications have been done away with. The summations must again

take place over selected ordinates, the values of which are given in table M.
The reflection factor ε_λ of the coloured surface is found by applying equation (24) to the normal white surface for which all values ε_λ are equal to λ, therefore:

$$Y_W = \tfrac{1}{30}\, Y_A \,.\, 30 = Y_A \text{ and } \varepsilon = Y : Y_W = \tfrac{1}{30} \sum_Y \varepsilon_\lambda$$

(hence ε is the average of the 30 values that ε_λ assumes for the wavelengths λ_y). Tables N and O give the selected ordinates for standard illuminants B and C.

§ 52 *Examples (continued)*

We should like to illustrate the use of the selected ordinates by two examples of calculation.
The first concerns the trichromatic coefficients of standard illuminant A. These have already been computed in section 37 in the "old" manner. The "new" calculation is given in table 5. For each selected wavelength of table L the value of E_A (see table F) has been tabulated. Addition then shows (according to equation 3):

$$X_A : Y_A : Z_A = 3326 : 3027 : 1075$$

or

$$x_A = 0.448,\ y_A = 0.4075,\ z_A = 0.145.$$

The second example concerns the colour of the green paper "nc 79" under standard illuminant A. This example was worked out in section 39 in the "old" manner. The new calculation is carried out in table 6. For each selected ordinate of table M the value of ε_λ (see fig. 28) has been given. Addition gives the following results:

$$\sum_X \varepsilon_\lambda = 4.351; \quad \sum_Y \varepsilon_\lambda = 6.691; \quad \sum_Z \varepsilon_\lambda = 9.508.$$

If we apply equation (25) with the values worked out in the previous example for X_A, Y_A and Z_A we find:

$$X : Y : Z = 14490 : 20220 : 10210$$

or

$$x = 0.3225, \quad y = 0.4545 \text{ and } z = 0.228.$$

For the reflection factor ε we find $\varepsilon = \frac{1}{30} \times 6.691 = 0.223$.
The results agree well with the results obtained in section 39 by another method.
We have always worked with 30 selected ordinates. The choice of this number is determined by two factors: the accuracy we wish to attain and the question whether the spectral distribution or the spectral reflection curve has a fairly level course.

In by far the majority of the cases occurring in practice the number 30 seems to be very satisfactory.

If the curves for E_λ or ε_λ are very flat, or if we are content with a fairly rough estimation of the colour point, 10 selected ordinates are sufficient. We take for these the wavelengths numbered 2, 5, 8, .. 29 in tables L-O (marked with an asterisk in the tables).

If on the other hand the E_λ and ε_λ have a very freakish course, or if we wish to attain a very great accuracy, it is sometimes desirable to take 100 instead of 30 wavelengths. H a r d y [3]) gives tables for this case.

n	TABLE 5			TABLE 6		
	$E(\lambda_x)$	$E(\lambda_y)$	$E(\lambda_z)$	$E(\lambda_x)$	$E(\lambda_y)$	$E(\lambda_z)$
1	22.3	42.7	17.7	0.250	0.477	0.144
2	27.2	55.7	20.9	0.449	0.476	0.162
3	31.6	62.7	22.5	0.313	0.444	0.177
4	36.7	67.4	23.7	0.263	0.415	0.191
5	44.0	71.2	24.7	0.228	0.390	0.205
6	80.2	74.4	25.6	0.204	0.362	0.219
7	89.3	77.4	26.6	0.185	0.337	0.236
8	95.3	80.2	27.5	0.170	0.314	0.248
9	100.0	82.9	28.1	0.157	0.294	0.260
10	103.9	85.5	29.2	0.147	0.275	0.271
11	107.4	88.1	30.0	0.137	0.257	0.280
12	110.6	90.7	30.9	0.129	0.239	0.288
13	113.6	93.1	31.8	0.122	0.223	0.295
14	116.5	95.7	32.7	0.115	0.207	0.301
15	119.1	98.1	33.6	0.110	0.195	0.307
16	121.7	100.7	34.6	0.107	0.180	0.310
17	124.2	103.3	35.5	0.104	0.169	0.316
18	126.6	106.0	36.5	0.101	0.157	0.322
19	129.8	108.6	37.5	0.098	0.146	0.330
20	131.6	111.4	38.6	0.095	0.136	0.340
21	134.1	114.4	39.6	0.093	0.126	0.351
22	136.6	117.4	40.8	0.091	0.118	0.368
23	139.3	120.6	42.1	0.090	0.110	0.382
24	142.1	124.1	43.6	0.088	0.105	0.416
25	145.1	127.9	45.1	0.086	0.100	0.449
26	148.5	131.2	47.1	0.085	0.095	0.476
27	152.4	137.1	49.6	0.084	0.090	0.494
28	157.1	143.0	52.9	0.082	0.087	0.493
29	163.6	150.8	58.0	0.082	0.085	0.474
30	175.6	164.8	68.4	0.086	0.082	0.403
	3326	3027	1075	4.351	6.691	9.508

n = number of selected wavelength.

For some light sources other than E, A, B and C selected ordinates have been calculated by M a c A d a m [3]) and R i c h t e r [7]). See also B o w d i t c h [1]). If we wish to determine the colour points of coloured objects for illumination by a light source for which we have no table of selected wavelengths, there are two ways open to us:

a. We can calculate such a table for ourselves from the values $E_\lambda \overline{X}_\lambda$, $E_\lambda \overline{Y}_\lambda$ and $E_\lambda \overline{Z}_\lambda$. This method is only to be recommended when we wish to examine a very large number of colours by the *same* light source.

b. In the formulae $X = \Sigma \varepsilon_\lambda E_\lambda \overline{X}_\lambda$, $Y = \Sigma \varepsilon_\lambda E_\lambda \overline{Y}_\lambda$ and $Z = \Sigma \varepsilon_\lambda E_\lambda \overline{Z}_\lambda$ we can calculate the products $\varepsilon_\lambda E_\lambda$ and sum them up over the selected ordinates of table L. In this way we save two-thirds of the otherwise necessary multiplications.

If we have to determine the colour point of a large number of only slightly differing lights we can take one of them as standard (spectral distribution E_λ) and determine photometrically the ratio f_λ for each other light source

of the power that this light source emits at the wavelength, to E_λ. For the required colour point the following then obtains:

$$X : Y : Z = \Sigma f_\lambda E_\lambda \overline{X}_\lambda : \Sigma f_\lambda E_\lambda \overline{Y}_\lambda : \Sigma f_\lambda E_\lambda \overline{Z}_\lambda,$$

so that we can determine the ratio of these sums by adding the values of f_λ for the selected ordinates calculated for the standard employed.
As f_λ will only change slightly with the wavelength, we can manage with a small number of selected ordinates. The method of selected ordinates can be applied in numerous other cases. We may mention here only their use for calculating brightnesses according to $B = M\Sigma E_\lambda V_\lambda$.
For colour calculation the method in the form dealt with here was first proposed by Hardy [2], but we find the principle in a somewhat modified form in Luther [1], while the well-known Rousseau diagram in photometry is really based on the same principle.

Finally we wish to point out the existence of methods that still further reduce the calculation involved in working out spectral measurements. Various ingenious instruments have been constructed — partly of an optical and partly of a purely mechanical nature — which perform the required multiplications and additions automatically. For further particulars we refer to the publications of Hardy [1], Rösch [2], v. d. Akker [2], Razek [1], Sears [1], Swank [1] etc.

§ 53 *The analogy of the eye and a combination of three photo-electric cells*

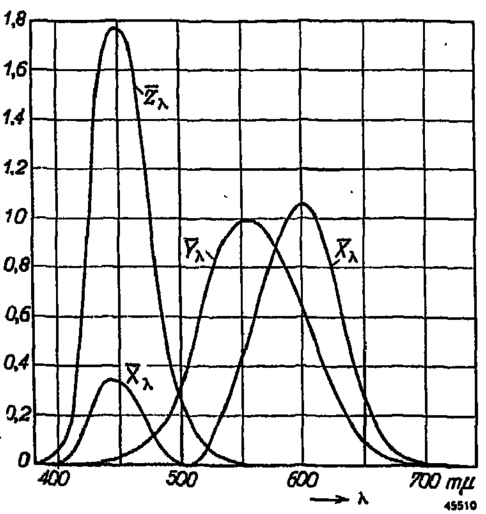


Fig 63
The eye may be replaced by three photo-electric cells whose relative spectral sensitivities correspond to the distribution curves $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$ of the equal energy spectrum.

In sect. 8 we pointed out that the properties of the eye in the matter of colour mixture could be perfectly explained if we accept the fact that each cone consists of a combination of three photo-electric cells. Let us now consider further this possibility of replacing the eye by photo-electric cells. Imagine three different photo-electric cells of which the spectral sensitivity is proportional to $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$ (*fig. 63* and table D). This means that, when we allow the same power of a

spectral colour of the wavelength λ_1 and of the wavelength λ_2 to fall in succession on the photo-electric cells, the electric currents produced in these two cases by the first cell will be in the ratio of $\overline{X}_{\lambda_1} : \overline{X}_{\lambda_2}$, while for the second and third cells the currents will be in the ratio of $\overline{Y}_{\lambda_1} : \overline{Y}_{\lambda_2}$ and $\overline{Z}_{\lambda_1} : \overline{Z}_{\lambda_2}$.

Further we suppose the photo-electric cells to be linear (*i.e.* that the current for a certain light is proportional to the power in the light beam) and that their action is *additive* (the current produced by a mixture is equal to the sum of the currents that each component would produce separately).

If we now direct a light pencil on these photo-electric cells having a spectral composition E_λ, the cells then produce the following currents:

$$\Sigma E_\lambda \overline{X}_\lambda = X; \quad \Sigma E_\lambda \overline{Y}_\lambda = Y; \quad \Sigma E_\lambda \overline{Z}_\lambda = Z.$$

The currents are therefore proportional to the trichromatic coordinates XYZ of the light directed onto the cells.

From this important result two conclusions can be drawn:

a. The three photo-electric cells act in the same way as the normal eye. For if the corresponding trichromatic coordinates of two pencils of light are equal, the eye sees no difference, while the one light pencil produces the same triplet of currents as the other. If, on the other hand, the eye does see a difference between the two light sensations, this means that at least one of the coordinates is unequal. In this case at least one of the photo-electric cells will also produce different currents for the two pencils of light.

As the whole colour comparison is always based on the judgment as to whether two light pencils produce an equal colour sensation or not, and the combination of photo-electric cells produces the same results as the eye in that respect, we can indeed say that we can unreservedly substitute the three photo-electric cells for the eye in the judgment of colour.

We might have used the verdict "the eye acts as a combination of three linear photo-electric cells with sensitivities $\overline{X}_\lambda$, $\overline{Y}_\lambda$, $\overline{Z}_\lambda$" as the starting point of the theory of the ordering of colour sensations in the chapters III-V. Grassmann's laws are already included in the supposed properties of the photo-electric cells, as we can easily prove. This way would undoubtedly lead to a shorter development than the considerations given in chapters IV-V, but we should have gained no insight into the experimental basis on which the whole rests.

Even if the photo-electric cells had the property of producing currents proportional to the trichromatic coordinates of another system, the combination would still be equivalent to the eye. This, for instance, would be the case if the cells had the sensitivities $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ (see table B) or $\overline{R}_\lambda$, $\overline{G}_\lambda$ and $\overline{B}_\lambda$ (table C). More generally speaking "the eye acts as a

combination of three linear photo-electric cells whose spectral sensitivities are linear combinations of $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$". In practice we shall only be able to realize such a trio of photo-electric cells if this "other" system gives exclusively positive coordinates.

b. From the proportionality of the currents produced and the trichromatic coordinates it further follows that we can also employ the three photo-electric cells to measure the X, Y and Z of a light beam. As we have eliminated the use of the eye in such a measurement (namely replaced it by the equivalent combination of photo-electric cells) we see here a typical example of the transition from a subjective to an objective method.

§ 54 *Objective colorimetry with the "photo-electric tri-colorimeter"*

This method is based on the facts described in section 53. Historically the method arose from one of the most primitive attempts to define colour sensations. The light source (or the coloured object) was examined in succession through red, green and blue tinted glass and the observed brightnesses were measured visually. The three numbers — if necessary multiplied by constants a, b and c — would produce the trichromatic coordinates. Here we have in fact the case dealt with in section 53, in which instead of the photo-electric cells the eye was used, covered by filters. Just as with the use of photo-electric cells, we also require the spectral sensitivity of the eye covered with filters to be proportional to the values $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$.
Now the spectral sensitivity of the eye itself is V_λ. If we look through a filter that transmits the value $(\tau_1)_\lambda$ for a wavelength λ, the eye thus reinforced has a spectral sensitivity of $V_\lambda \cdot (\tau_1)_\lambda$. The same with the use of the two other filters : $V_\lambda \cdot (\tau_2)_\lambda$ and $V_\lambda \cdot (\tau_3)_\lambda$. The required equivalence with $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$ therefore exists only if the filters comply with the following conditions:

$$V_\lambda \cdot (\tau_1)_\lambda = c\overline{X}_\lambda; \quad V_\lambda \cdot (\tau_2)_\lambda = c\overline{Y}_\lambda; \quad V_\lambda \cdot (\tau_3)_\lambda = c\overline{Z}_\lambda \qquad \text{or}$$

$$(\tau_1)_\lambda = c\overline{X}_\lambda : V_\lambda; \quad (\tau_2)_\lambda = c\overline{Y}_\lambda : V_\lambda; \quad (\tau_3)_\lambda = c\overline{Z}_\lambda : V_\lambda \qquad (26)$$

If the filters do not comply with this requirement there is the risk that two light beams with equal trichromatic coordinates will yet produce different results.

This risk is eliminated if the filters satisfy the more general requirement arising from (26) by replacing $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$ by arbitrary linear combinations of these three functions.

But even if the filters do come up to the requirement — which was first explicitly stated by Luther[1]) — the method has still

the great disadvantage that visual brightness comparisons have to be made in which there is always a colour difference between the two halves of the field of vision. To what uncertainties this may lead we have already seen in section 11.

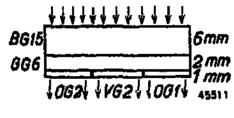


Fig. 64
A combined filter as used by Dresler for adapting the photocell sensitivity to the curve $\overline{Y}_\lambda$.

The only way to avoid these uncertainties is by substituting an objective measurement for a subjective one, *i.e.* by employing three photo-electric cells instead of the eye.

The great practical difficulty (which by the way exists for the subjective method too) is to comply with the requirement that the spectral sensitivity of the photo-electric cells must be proportional to the quantities $\overline{X}_\lambda$, $\overline{Y}_\lambda$, $\overline{Z}_\lambda$. No single type of photo-electric cell fulfils such requirements unconditionally.

As a rule we employ one photo-electric cell to which we endeavour to give the required spectral sensitivities by using different filters in succession. This sounds very simple but in practice we encounter very great difficulties if we wish to satisfy the requirements with accuracy. The filter that has the task of adding sensitivity $\overline{X}_\lambda$ to the photo-electric cell, owing to the queer shape of $\overline{X}_\lambda$ (see fig. 63), is particularly difficult to achieve. Efforts have been made to remove these objections by ingenious combinations of filters used partly in series (the light passing the filters in succession) and partly in parallel. The latter principle, which is particularly freely employed by Dresler[2]), is in fact equivalent to the employment of one or two different photo-electric cells used parallel to each other [Barnes[1])]

In fig. 63 it strikes one that the short-waved part of $\overline{X}_\lambda$ is about proportional to $\overline{Z}_\lambda$. This was made use of by employing the same filter for both parts [v. d. Akker[1]), Dresler[2])].

Fig. 64 illustrates the use of filters placed one behind the other or next to each other. It represents the filter used by Dresler[2]) to give a spectral sensitivity to the photo-electric cell proportional to $\overline{Y}_\lambda$, and it is constructed entirely from the well-known Schott filters.

We can also use the fact that it is sufficient for spectral sensitivities to be proportional to arbitrary linear combinations of $\overline{X}_\lambda$, $\overline{Y}_\lambda$ and $\overline{Z}_\lambda$. In this way we can arrive at filters easier to realize [Winch[1])]. A principle suggested a long while ago by Ives[4]), i.e. to analyse the light spectrally and to bring about the necessary reduction of the various spectral colours in the spectrum by templates, has been followed on a limited scale both in photometry [Voogd[1]), H. König[1]), Laporte[1]), Winch[2])], and in colorimetry [Knipe[1]), H. König[5])]. See also Guild[8]), Hunter[1]), Perry[1]).

Summing up we can make the following judgment of "photo-electric colorimetry". Of all methods it is perhaps the one with the best chance of survival, as it combines relative simplicity with very rapid measurement, and with all the advantages that an objective measurement offers over a subjective one. Undoubtedly there are still technical difficulties to be overcome. At the moment. the same accuracy can be attained with relatively simple means as with the best subjective methods. If we should succeed in raising the accuracy to the level of that of the best spectro-photometric methods — and this is chiefly a question of better filters — then probably no single method would be able to compete against that of photo-electric tri-colorimetry.

CHAPTER IX

Subjective colorimetry

§ 55 *Range of application and classification of subjective methods*

In section 49 we have already dealt with the distinction between objective and subjective methods of colorimetry and the most important advantages and disadvantages attached to both systems. In the past, of course, the subjective methods had everything their own way. The knowledge of the properties of the eye, research into the laws of colour mixture and into the existence of normal and abnormal organs of sight, the principles upon which the C.I.E. system was based, were all built upon subjective measurements. But subjective methods are still of importance to-day, for scientific purposes: for continued research into the properties of the eye and for eventual improvements of the standard system of the C.I.E., and for practical purposes: in all cases in which the advantages of the subjective method are of greater weight than the disadvantages. The choice between subjective and objective methods is determined in a great measure by the accuracy we wish to attain. In this respect we can distinguish the following cases:

a. The maximum accuracy is required without much regard to cost in time and money. In this case objective methods are the most suitable. It is possible to limit the errors made in determining the XYZ values so that they are no greater than the colour differences that the eye can just distinguish, while in comparing colours of the same species and in measuring small colour differences a considerably greater accuracy can be reached.
b. No greater accuracy is desired than can be attained with the best visual instruments. In this case both groups of methods are satisfactory and the choice made will often depend on all kinds of side issues.

   "Photo-electric colorimetry" will in future probably be of particular importance here (section 54). The mutual differences between measurements made with the various instruments and

by observers and between the measured results and the C.I.E. system in most cases do not exceed more than a few times the colour differences just visible to the eye. As for the establishment of small colour differences, the objective methods for case (b) are about as sensitive as the unaided eye.

c. If accuracy is of still less importance, or if it is sufficient to compare the colours with certain standards without the necessity of knowing the values of XYZ with any precision, we can use the simplest methods of subjective measurement. These are simpler than rough objective measurements.

Subjective measurements are always based on the fact that the colour sensation to be examined is made indistinguishable from another colour sensation. This "other" colour sensation can arise in various ways. In the first place, and this is especially important for the judgment of coloured surfaces, we can select from a large collection of coloured objects the one whose colour most resembles the colour to be examined. This method, seemingly so simple (section 56), is only of use if accuracy is of little importance (case c).

In the second place we can *match* the colour sensation by additive mixing of colours (sect. 6) or by varying a known colour in some other known way so that an equality of colour sensation is produced. With the aid of instruments by which such a matching is brought about we can attain the degree of accuracy mentioned in case (b).

§ 56 *Colour determination with the aid of a colour atlas*

If we have at our disposal an extensive collection of colour samples (a colour atlas), it seems at first sight easy to determine the colour of a given object by finding out to which item of the collection the colour of the object to be examined corresponds. This would appear to be a very simple, direct, accurate time-saving method of colour determination.

In practice, however, it is disappointing. For the method to be satisfactory the colour atlas must be of very high quality — which is true of none of the existing colour atlases — and when using it various precautions must be taken with which hardly any user ever complies. If it is desired to use a colour atlas to determine the colour of a given object with some accuracy, the atlas must comply, *inter alia*, with the following conditions:

1. The atlas must be complete, *i.e.* contain all colours met with in

practice. The difficulty of this can be understood when we remember that not only must it possess representatives of all colour points of a great part of the colour triangle, but that these colours must also be present in various brightnesses. In other words, we must have a whole series of colour samples at hand each having the same chromaticity but varying reflection factors, from values so low that the colour can hardly be distinguished from black, up to the highest values that can be realized (see sect. 46).

2. The atlas must be sufficiently dense, *i.e.* two neighbouring colours must differ very little from each other. The ideal case would be that two neighbouring colours differed so little from each other that they could barely be distinguished by the eye. For accurate colorimetry the colour differences between neighbouring examples must certainly nowhere exceed more than twice this amount. The first and second conditions imply that the atlas must contain an enormous number of specimens. As an illustration Ostwald's colour atlas may be mentioned, which contains in its original form more than 2000 coloured cards and still does not comply either with the first or with the second condition!

   Ames[1]) estimates that about 13000 cards are required to fulfil conditions 1 and 2. The number of surface colours distinguishable with a given illumination under favourable conditions of observation is very much greater: according to an estimation by Judd[13]) about ten million!

3. If we wish to employ this method to define colours in the XYZ system, the X, Y and Z values assumed under a certain illumination (preferably two illuminations, for instance A and B) must be marked on each specimen of the collection.
4. Even if we do not arrange the colours to be measured in the XYZ system and are satisfied with giving the number of the colour in the atlas used, we must ask for reproducibility. Different copies of the atlas must be identical to the extent that it does not matter if we have a colour measured by P or Q (supposing they both possess an Ostwald atlas).
5. The condition of *colour fastness* implies that in normal use the colours shall not change in the course of time. This condition is particularly important if we wish to measure the spectral reflection curve of a specimen. For this purpose we generally require a fairly protracted and strong illumination.

In using an atlas for the classification of unknown colours the following rules must be observed. In the first place both the colour to be measured and the comparison colours must be illuminated by a well

defined light source which must be quoted with the result For it may happen that two coloured objects under incandescent light give the same colour sensation but in daylight show clear differences In the second place we must consider how the samples are illuminated If possible the instructions given in section 38 (fig 27) should be adhered to The use of other illuminating and observing arrangements should be quoted with the results This rule is in some cases of the greatest importance, as for some surfaces (especially with textiles) the colour and the reflection factor may depend to a fairly large degree on the directions of illumination and observation Thirdly, we must try to observe the colours in such a way that nothing can be seen of the structure of the surface, therefore in such a way that we cannot see of what kinds of material the two colours to be compared are made We can attain this, for instance, by placing the object far enough away, or by looking at the surface through a lens, so that it is not sharply reproduced on the retina We can also cause the structure to disappear by letting the surface revolve rapidly If we do not take such precautions great difficulties will be experienced in colour comparison when one of the objects has a surface that is not quite smooth and uniform in colour (for instance, textiles, wood, agricultural products etc.). Finally we must comply — at least approximately — with the conditions summed up in section 19, which must apply to all visual colour measurement

Only when all the requirements mentioned have been fulfilled and all precautions taken, can this method develop into one of precision Since, however, in practice these conditions have never been satisfied and are partly impossible to satisfy, colour determination with the aid of a colour atlas must be classed among the rough estimation methods (group (c) of section 55)

We by no means wish to imply, however, that for certain purposes the method would not be of great practical value.

Some well-known colour atlases are those of Ridgway[1]), Munsell[1]), Ostwald[1, 2]), Maerz[1] and Prase [Baumann[1])]. See further the surveys by Ames[1]), Pander[1]), Richter[9]) and the extensive publications regarding the Munsell system widely known in America (for instance, Nickerson[3]), Granville[1, 2]), Gibson[6]), Glenn[1]), Kelly[1]), Newhall[2, 3]), Judd[14]), Tyler[1]), Bond[1]) etc.].

§ 57 *Prisms often used in trichromatic colorimeters*

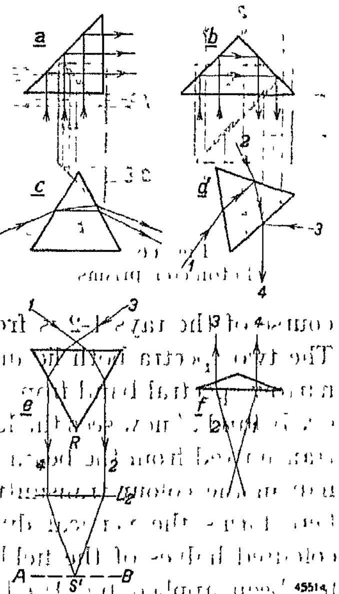


Fig. 65
Application of prisms in trichromatic colorimeters. The parallel ribs of the prisms are to be imagined perpendicularly to the plane of drawing (see fig. 40)

We wish to discuss now a large group of subjective methods by which a colour sensation is built up from a very limited number of colours, to be indistinguishable from the colour to be examined. In the majority of these methods this reproduction takes place by additive mixture of three fixed colours. In such cases the instruments are called "trichromatic colorimeters". The principle of this method, on which the whole trichromatic theory is actually based, has been repeatedly discussed, so that we can restrict ourselves here to the discussion of the construction of measuring instruments based on this principle. Rather than sum up an endless row of instruments and methods of execution, we shall just mention the various parts that occur in all such apparatuses. For the different techniques the reader is referred to the extensive literature on the subject. An acquaintance with the various parts will facilitate this study. Every trichromatic colorimeter contains one or more kinds of prisms which are used for the most divergent purposes. A number of the most frequently occurring types, and the manner in which they influence the passage of light rays are given below.

The simplest prism is one with a *rectangular* isosceles transverse section used almost exclusively as a mirror (*fig. 65a, b*). The rays striking one of the sides at an angle of 45° are totally reflected. The entrance and exit of the light always takes place perpendicular to the surface, so that no undesirable colour dispersion takes place. The three-sided prism with *equilateral* transverse section is employed for the spectral analysis of light (*fig. 65c*, see also fig. 60) and for the additive mixture of lights (*fig. 65d*). Here lights (1) (2) and (3) are mixed and leave the prism as one pencil of rays (4). This method can only be applied if (1) and (2) are spectral lights, otherwise an undesirable colour displacement takes place.

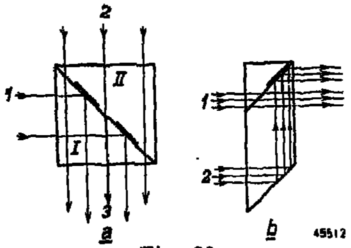


Fig. 66
Photometer prisms.

Finally the equilateral triangular prism occurs in the combined rôle of bringing about the spectral analysis of light and forming a field of vision divided into two halves for colour comparison. *Fig. 65e* illustrates this. We have here actually a combination of two spectroscopes of the type sketched in fig. 60 having a common prism. The course of the rays 1-2 is from one opening and 3-4 from the other. The two spectra both lie in the plane AB, so that we can cut out a narrow spectral band from both by placing a common slit at S′. An eye behind S′ now sees the left half of the field of vision, in the colour transmitted from the beam 3-4 through the opening S′, and the right half in the colour transmitted by beam 1-2 through S′. The rib R here forms the vertical dividing line between the two differently coloured halves of the field of vision. The principle described here has been applied by Helmholtz for the comparison of light mixtures consisting each of two spectral colours (see section 72). The bi-prism *(fig. 65f)*, a three cornered prism the transverse section of which is an isosceles triangle with very acute angles on the hypotenuse, serves to turn two divergent beams 1 and 2 into the parallel beams 3 and 4. As the part of beam 1 falling on the left half of the bi-prism leaves the prism again at a strongly divergent angle, an eye placed at some distance sees on the left-hand side of the prism only light from beam 2 and from the right only light from beam 1. The rib R therefore again forms a vertical dividing line between two differently coloured halves of the field of vision.

Slightly more complicated are the so-called photometer prisms of which fig. 66 gives two examples. The prism of *fig. 66a*, the Lummer-Brodhun prism, is to be considered as a combination of two prisms I and II each being shaped as a half cube. Before they are stuck together a part of the oblique side of one of them is made into a mirror (the part heavily drawn in fig. 66a). *) After sticking the two parts together and directing two beams onto it in the way indicated, we see in direction 3: from the mirrored parts of the diagonal light from beam 1 and from the unmirrored parts light from beam 2. By giving a special shape to the mirrored parts we can divide the field of vision into any parts

*) This can be done by grinding away a part of the dividing surface.

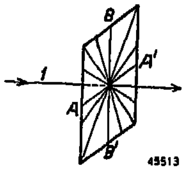


Fig. 67
Polarized and unpolarized light.

radiating partly light 1 and partly light 2. The most usual is again a vertical division into two halves. Such a prism is used in photometry to adjust the two halves of the field of vision to equal brightness, while it is employed in colorimetry to adjust two beams to an equal colour. Another division of the field of vision is into a number of vertical bands giving in turn the colours of the two pencils of light (fig. 66a). *Fig. 66b* shows a photometric prism of a slightly different construction giving a field of vision illuminated in one half by light 1 and in the other half by light 2 (the part of the boundary surface drawn heavily is again a mirror).

Finally *fig. 68* shows some prisms whose action is based on the properties of polarized light. Considering a beam 1 *(fig. 67)*, we see a vibration phenomenon propagating itself along this line. The direction of the vibrations is always perpendicular to the direction 1. Usually all vibration directions lying in the plane ABA'B' are represented, but in certain cases a pencil of rays is produced from 1 (by the action of prisms, mirrors, etc.) in which only one direction of vibration occurs (for instance the direction AA'). In this case we say that the light is polarized. If another ray only contains vibrations in the direction BB' we then say that the plane of polarization of the first ray is perpendicular to that of the second, or that "the two rays of light are mutually perpendicularly polarized."

The prisms shown in fig. 68 each produce two such mutually perpendicularly polarized light beams, which also differ in other respects, by which they can be distinguished. In contradiction to the prisms illustrated in figures 65 and 66, which are usually made of glass, the phenomena sketched in fig. 68 only occur in prisms cut in a certain way from the crystals of substances such as quartz or Iceland spar. A ray of light falling on such a prism (fig. 68a) is split in the manner indicated into two beams (2 and 3) which are refracted in different directions and which appear to be mutually perpendicularly polarized; such a prism is called "birefringent". Since in the case of fig. 68a the two beams leaving the prism are parallel, this form of prism is not suitable for separating these beams completely. With a prism of

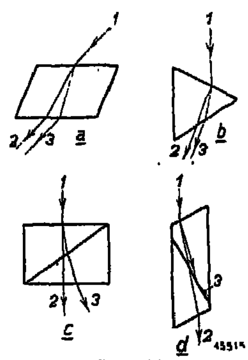


Fig. 68
The use of polarized light in trichromatic colorimeters.

the type sketched in *fig* 68*b* this could be done, but it causes an undesirable colour dispersion, while the fact that the direction of both departing beams 2 and 3 differs from that of 1 can also be very annoying Both objections are removed by using the prism of *fig.* 68*c*. This consists of a bi-refringent prism I and a glass prism II (the lower in the figure) having for beam 2 the same refractive index as I. Beam 2 now leaves in a direction parallel to 1 and shows no colour dispersion whatsoever. We can obtain the same result by taking instead of II a bi-refringent prism of Iceland spar ground in a different manner from I We can also arrange it so that the beams 2 and 3 lie symmetrically with respect to direction 1 (W o l l a s t o n prism) A W o l l a s t o n prism is used if for some reason or other we wish to split a beam into two mutually perpendicular polarized beams, and also (and now we change the direction of the arrows of fig 68c) to unite two beams of different direction into a light pencil in which the rays proceeding from 2 and 3 are still mutually perpendicularly polarized.

If it is required to split light into two mutually perpendicular polarized beams and only use one of those beams, then the prism of *fig.* 67*d*, the Nicol prism, commonly called the "nicol" for short, is used. Here one polarized beam (2) leaves the prism in the same direction as (1) while the other pencil (3) is totally reflected at the boundary surface of I and II and then reaches the boundary of II to be completely absorbed there. If light pencil 2 is again directed onto a nicol identical with the first one and in the same position, the second nicol will let the light pass unimpeded. If the second nicol is now turned 90° on its axis (parallel to 2), no light whatever will appear, all the light of 2 has been reflected and absorbed in the manner of 3 If the second nicol is in an intermediate position the same kind of division takes place as in the first nicol, so that a part of 2 again leaves the prism parallel to the axis and another part is destroyed If the second nicol is turned slowly about its axis the quantity of light transmitted will gradually decrease from its maximum value (the two nicols parallel) to zero (crossed nicols). In the combination of the two nicols we have therefore a simple method of weakening a pencil of rays in a measurable manner If we observe the position of the second nicol we can calculate the reduction factor quite simply

A nicol can also be used to vary the ratio of the two components in additive mixture For this purpose care is taken that the two components are mutually perpendicularly polarized (this can be done with a W o l l a s t o n prism). If this mixture is directed onto a nicol the intensity ratio of the two components will change continuously when the nicol is turned. In

particular two mutually perpendicular positions of the nicol can be found in which one or the other component is transmitted exclusively. In certain cases polarized light must be used with care for intensity regulation, particularly when the reflection factors of a surface are dependent on the direction of polarization of the light [i.e. sonic textiles; see Nutting[1])]

§ 58. *The choice of primary colours.*

We shall now consider step by step the problems met with in the construction of the trichromatic colorimeter. In the first place we have the choice of the primary colours from which the matching mixture will be built up. Sometimes three spectral colours are selected for this purpose, or they may, for instance, be obtained by placing three different coloured filters in front of incandescent lamps (we then speak of "filter colours"). In either case it is important that the three primary colours lie as far apart as possible. In other words, their colour points must form as large a triangle as possible in the colour plane. It would appear then that the great majority of colours can be measured without having to resort to negative quantities (section 21).

In the oldest trichromatic colorimeters, which were chiefly built for the purpose of studying the laws of quantitative colour mixture, spectral primary colours were practically always used; as in this case the calculation of the results is simplest [Maxwell[1]), Helmholtz-König[9])]. Subsequently, however, when a deeper insight into theory was gained, the advantage was not felt so keenly and colorimeters with composite primary colours were used for the same purpose [Guild[1])] as well as instruments with spectral colours [Wright[2])].

The situation has altered slightly since the introduction of the *XYZ* system, when the demand arose that the results obtained with trichromatic colorimeters should correspond with this system as far as possible. This resulted in a great disadvantage accompanying the use of spectral primary colours, for if we reproduce the colour sensations to be measured by mixing three spectral colours, the two matching mixtures as a rule differ strongly from each other in their spectral compositions. These great divergences cause the results to be greatly influenced by the unavoidable individual differences in the colour vision of the various observers. With the use of primary colours, each containing wavelengths from a great range of the spectrum, the difference in spectral composition is much smaller, so that the results of the various observers deviate less and it is easier

to link up well with the XYZ system. After the instruments designed by Ives[1]), Wright[1]) and Verbeek[1]), which still worked with spectral primary colours, we find filter colours used almost exclusively [Donaldson[1]), Guild[2]), Richter[5]) etc.] in modern colorimeters.

In visual photometry, too, the same trend can be seen. Before we make a visual adjustment (for instance with a flicker photometer) we must try to equalise, by means of filters, not only the colours of the two lights to be compared but also their spectral composition.

§ 59 *Additive mixture of primary colours*

When the primary colours have been chosen they must be additively mixed. Various means can be used. One of the oldest methods is with the aid of a spectroscope (*fig. 69a*, see also fig. 60b). Light from an incandescent lamp falls on an opening S and produces a spectrum in the plane AA′ via prism P and lenses L_1 and L_2, so that the light transmitted through opening S consists of only one particular spectral colour.
If we now add a second opening S_2 a spectrum will also be produced in the plane AA′, but this will have been displaced with respect to the first spectrum. The result is that a spectral colour of another wavelength will fall on the opening S′. If we use the openings S_1 and S_2 simultaneously an additive mixture of the two spectral colours will therefore appear at S′.
This method of mixing primary colours can of course only be applied to spectral primaries. Amongst others it is used in the early instruments of Maxwell and Helmholtz. *Fig. 69b* illustrates a very simple method of mixing colours additively. The three projection lanterns 1, 2 and 3 project three different coloured light spots on the screen AA′. If these spots cover each other an additive blended colour is produced. This method is much used for demonstration tests but is

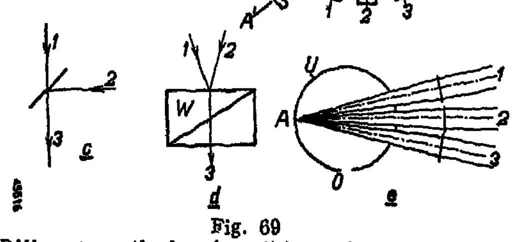


Fig. 69
Different methods of realising additive mixing (partly after Richter).

hardly ever employed for the measurement of colours. A more favoured method is shown in *fig. 69c.* In this the oblique line represents a glass plate which partly reflects the beam 2 in direction 3 and at the same time partly transmits beam 1 without change of direction. Beam 3 therefore contains the additive mixture of 1 and 2. If another colour is to be added (4) a second mirror which mixes beams 3 and 4 is used in the same way.

Fig. 69d shows a method of mixing that has already been discussed in section 57. By the aid of a Wollaston prism W the two beams 1 and 2 are combined into beam 3, in which the components proceeding from 1 and 2 have been mutually perpendicularly polarized. This method is not suitable for blending three or more colours.

Finally *fig. 69e* shows a method, adopted in various modern instruments [Donaldson[1]), Richter[5]) etc.], which employs a photometer sphere, also called an Ulbricht sphere (U). This is a sphere the inside of which has been covered with white paint which reflects the light well and diffusely. If we illuminate a small part of the inner surface the rest of the sphere is illuminated *quite uniformly* by the reflection of the light from A. In photometry this property is utilized to measure the total luminous flux radiated by a light source. The illuminant is suspended in the centre of the sphere and the brightness assumed by a part of the inner wall of the sphere as a result of the reflected light is proportional to the required total light flux, if care is taken that this part is not illuminated by direct radiation from the light source. In trichromatic colorimeters the various coloured beams 1, 2 and 3 are concentrated onto a small part A. If we observe the opposite part of the wall of the sphere through an opening O, we see the additively mixed colour radiated with uniform brightness. Compared with fig. 69b this method has the advantage that all difficulties that may occur in fig. 69b (irregular brightness, the three light spots not accurately covering each other, coloured edges etc.) are automatically avoided. It must be noted that in photometry very large spheres (viz. 2 metres in diameter) are usually employed, as the dimensions of the sphere must be large compared to those of the light source. In the trichromatic colorimeters, however, small spheres are used (about 10 or 20 cm diameter) which have the advantage of producing adequate brightnesses of the wall of the sphere even with a relatively small amount of light. In the methods of mixture dealt with so far the light of the three primary colours was united in one pencil of rays, so that the same part of the retina received the three lights simultaneously.

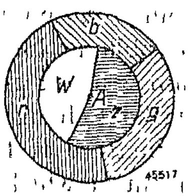


Fig 70
Additive mixing with the sectored disc (z = black)

Another method of additive mixing consists of directing different coloured lights in succession onto the eye with such a high speed that a constant colour sensation is obtained; blended colours are then produced which obey the same laws as the colours produced by simultaneous additive mixing For this reason the method of swift colour exchange is also useful for mixing primary colours

Fig 70 shows the oldest and best known method of applying this principle, one which is often used for demonstration tests and which allows a blending of the colours proceeding from different coloured objects (for instance papers) The colours are placed on a flat disc (in our example red, green, blue, black and white) By turning the disc with sufficient speed about the axis A (perpendicular to the drawing) we see the inner circle uniformly illuminated, while the outer ring is also seen as one colour

By a suitable choice of the surfaces observed with the various colours the dividing line between circle and ring can be made to disappear

In our example grey (from black and white) has been matched by a mixture of the three colours red, green and blue

In some colorimeters the radiations of three colour samples are actually used as primary colours [Nickerson [1])] and these are mixed in a similar manner Instead of a revolving disc a stationary disc is used, observed through a revolving prism.

In order to mix coloured light rays we employ a slightly different method (*fig 71a*). A system of two prisms A and A' is made to revolve rapidly about the axis OO' The light rays then follow the path drawn In the fixed dividing wall CD there are openings provided with coloured filters which transmit different coloured rays in succession to certain positions of the prisms If the beam of light is broad and homogeneous enough (*fig 71b*) one prism is sufficient. *Fig 71c* is a sketch of the dividing plane CD; three coloured light rays are to be mixed; R, G and B are the openings provided with coloured filters; 1, 2 and 3 are adjustable shutters, closing part of the openings and allowing the ratio of the three colours to be regulated (light can only pass through the shaded parts in the figure)

The instrument sketched in fig 71 is known as the "Brodhun sector" and is employed in the Guild colorimeter, among others.

There are a few other possibilities of additive colour mixing but these are not applied in colorimetry.

Of these we may mention:

1. The additive mixing that takes place in the eye when looking at a sufficiently fine mosaic of different coloured spots (employed in colour photography and colour printing)
2. Binocular colour mixing. If different coloured lights fall on corresponding parts of the retina of each eye, in certain circumstances a mixed colour is seen. For this mixture other laws obtain than for "ordinary" additive colour mixing [Lohmann[1]), Trendelenburg[2]), Rochat[1]), Hecht[2]), Livshitz[1])]

§ 60 *Trichromatic colorimeters. Various considerations*

An important point in the construction of trichromatic colorimeters discussed below is the manner in which the intensities of the three primary colours are regulated so that the magnitude of the quantity of each primary colour used can be read directly from the regulating mechanism.

The most usual method of regulation in visual photometry, namely varying the distance of the light source, is very seldom employed here [see, however, the Priest[4]) colorimeter]. As a rule a fixed position for the luminants is preferred, and almost without exception the quantities of light are regulated when making the colour match by one of the following two methods:

a. By adjustable shutters or diaphragms that screen off a part of the diameter of the beam of light and thereby regulate the light flux transmitted. These diaphragms are of the most divergent shapes. The well-known iris diaphragm employed in cameras (an imitation of the diaphragm of the eye) is one of these. We have already seen quite another shape in fig. 71c. In using such diaphragms care must be taken that they are set up at the right place in the path of light, so that the movement of the diaphragm only produces a gradual alteration of the quantity of light without influencing either the shape or size of the field of vision [Guild[2]), Donaldson[1]), Richter[5]), Wright[1]), Verbeek[1]), etc.]

b. With nicols. This method we have discussed at the end of sect. 57. It is employed in the instruments of Helmholtz, Nutting[1]), Richter[5])

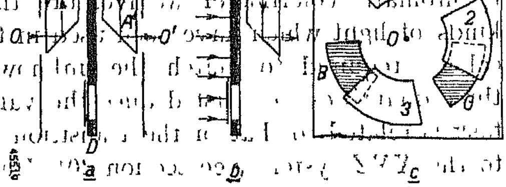


Fig. 71
Additive mixing with Brodhun sector

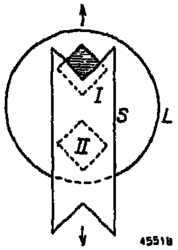


Fig. 72
Method for obtaining negative amounts of a primary ("optical switching over", according to Richter).

In order to produce a field of vision divided into two halves a Lummer Brodhun prism (fig. 66a) is commonly used. In some instruments other means are employed, such as the bi-prism (fig. 65f) or the prisms described in fig. 65c and fig. 66b. We shall now discuss briefly a few other problems arising in connection with trichromatic colorimeters. In all instruments the possibility exists of also producing negative quantities of one of the primary colours, that is to say, of adding a certain quantity of one of the primary colours to the colour to be measured instead of to the matching mixture.

A very ingenious method of attaining this end has been invented by Richter *(fig. 72)*. A screen is placed before a lens L with two square openings I and II. A queerly shaped shutter S, mobile in a vertical direction, always closes one of the openings entirely and the others partially. The light transmitted by I forms part of the matching mixture; the light proceeding from II follows another route and is finally added to the colour to be measured. In the position of fig. 72 only the shaded part of I makes a contribution to the matching mixture. If S is now shifted a little higher, this contribution becomes increasingly smaller. At the moment, however, when the contribution becomes zero (when I is quite closed) II begins to open and to produce a "negative" contribution. Without the observer noticing anything he has passed automatically from a positive contribution through zero to a negative contribution. [Richter[5]) speaks here of an "optical switch over"].

If we wish to use a colorimeter to measure the colour of a coloured object an apparatus must be added to illuminate the object in a particular, prescribed manner (for instance according to fig. 27). In some instruments [Richter [5])] this apparatus is built into the case in which the whole colorimeter is mounted. In others this apparatus is constructed separately from the instrument itself. With a trichromatic colorimeter we measure the quantities of the three kinds of light which have been used in the instrument as primary colours required to match the unknown colour K. From these three quantities we must deduce the values of X, Y and Z. This task is related to that of the transition from the $B_1B_2B_3$ system to the XYZ system (see section 30). Here too we began with the required quantities of the three kinds of light. If we know the XYZ

coordinates of the primary colours used we can deduce formulae, just as in section 30, enabling us to calculate the XYZ values of the colour K from the measured quantities $B'_1B'_2B'_3$. If

for unit brightness of the first primary colour:

$$X = a_1,\ Y_1 = b_1,\ Z = c_1,$$

for unit brightness of the second primary colour:

$$X = a_2,\ Y = b_2,\ Z = c_2,$$

for unit brightness of the third primary colour:

$$X = a_3,\ Y = b_3,\ Z = c_3,$$

and if colour K can be reproduced by a brightness B'_1 of the first, B'_2 of the second and B_3' of the third primary colour, then the XYZ values of the colour K are clearly:

$$\begin{aligned} X &= B_1'a_1 + B_2'a_2 + B_3'a_3, \\ Y &= B_1'b_1 + B_2'b_2 + B_3'b_3, \\ Z &= B_1'c_1 + B_2'c_2 + B_3'c_3. \end{aligned}$$

Here therefore are the required equations. They have a form similar to that of the transformation equations (9) of section 30. In order to be able to pass from the measured values to the XYZ values we need therefore know only the coefficients $a_1a_2 \ldots . c_3$ or the XYZ values of the three primary colours used. But these can be calculated with the aid of table D from the spectral distribution of the light source and the spectral transmission curves of the filters used.

Corrections must be made for the lack of selectivity of the other parts used in the instrument (mirrors, photometric spheres etc.). Another method given by Guild[1]) to determine the coefficients a_1-c_3 is by measuring a number of known light sources (for instance some spectral colours, a standard white) with the trichromatic colorimeter. Each measurement then supplies us with some equations for the coefficients sought for. If we make more of such measurements than are required for determining $a_1a_2 \ldots . c_3$, we have a check on the accuracy of the measurement and on the degree of correspondence of the observer with the XYZ system. This method is less intricate than the previous one but has the disadvantage that in determining the coefficients all the uncertainties of the subjective measurement slip in and with the spectral colours in this case to a far greater degree than with the normal use of the instrument.
The following method has been suggested by Smith[1]) and Guild to minimize the influence of individual differences and of small alterations in the instrument. Let the observer intending one day to make a series of measurements first measure the colour of a standard illuminant (B or C). From this measurement certain values $(B_1'B_2'B_3')$ must appear which can be calculated from (1). If the observer now obtains values which are respectively p_1% p_2% and p_3% too large, the further observations of that day can be corrected by reducing all other readings by p_1, p_2 and p_3%. If one of the p's is large there is something amiss either with the instrument or with the observer!

A special method is recommended by Wright [10]) in the event of having to measure a large number of only slightly differing colours. Instead of the standard illuminant a fixed colour (in the range of the colours to be measured) is taken, which has been established by an objective method.

There are still two methods worth mentioning which in certain cases facilitate the calculation of the coordinates of one system in terms of the other.

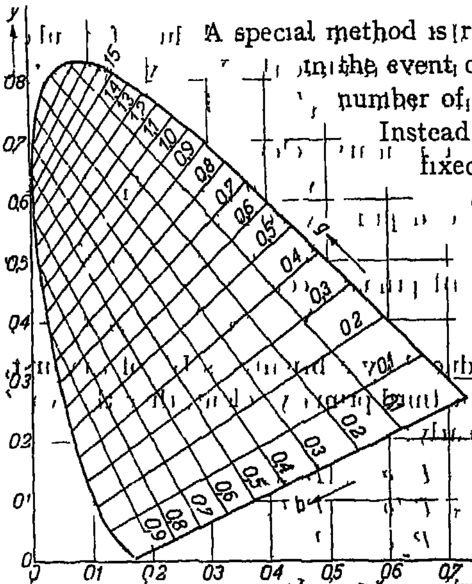


Fig. 73. Graphical determination of x and y from g and b.

1. *Graphical method.* If the data from any given system (RGB) are to be transformed in the XYZ system the lines of constant g and those of constant b are drawn in an x, y diagram. In *fig.* 73 this has been done for the (RGB) system of the C.I.E. dealt with in section 36. Such a figure can be used for transformations in both directions. For instance we can read directly from it that the standard white E has the coordinates $(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$ in both systems. In the construction of such figures use is made of the fact that all lines of constant g as well as all lines of constant b pass through one point (in the example chosen these two points are $x = +0.843$, $y = -0.3116$ and $x = -0.386$, $y = 1.365$) and that there is a system of parallel lines in the x,y-plane on which the lines of constant g and those of constant b make regular intercepts (in our example the lines

$$0.9146\, x - 0.0854\, y = \text{const.})$$

2. *Nomograms.*

If one takes (*fig.* 74) two vertical straight lines for the y and z scales and two horizontal straight lines for the g and b scales it is then possible to make a linear scale on each of those straight lines such that any diagonal runs exactly through a set of corresponding values of y, z, g and b. If we put the projective tranformation in the form

$$g = \frac{a_1 y + b_1 z + c_1}{a_3 y + b_3 z + c_3},$$

$$b = \frac{a_2 y + b_2 z + c_2}{a_3 y + b_3 z + c_3}$$

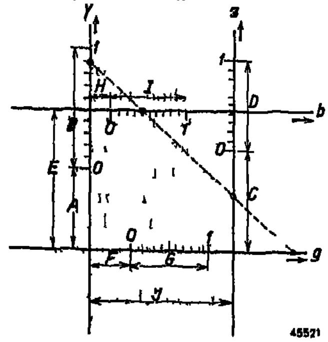


Fig. 74. Principle of nomographic method for carrying out a projective transformation in the colour plane.

and if we represent the subdeterminant of a_1 in $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ by α_1 then the following must obtain

$$A, B, C, D, E = a_3\alpha_2 - a_3\gamma_2 \quad b_3\beta_2 - b_3\gamma_2 \quad \frac{\Delta a_3 b_3}{\gamma_1} \text{ and}$$

$$F, G, H, I, J = b_1\gamma_1 - b_3\gamma_1 \quad b_2\gamma_2 - b_3\gamma_2 \quad \frac{\gamma_1\gamma_2}{a_3}$$

In fig 75 as an example the nomogram has been drawn for the transformation of RGB (section 36) into XYZ The line $g = b = y = z = \frac{1}{3}$ has been added The nomograms can be again used in both directions For other nomograms see H o l m e s [1])

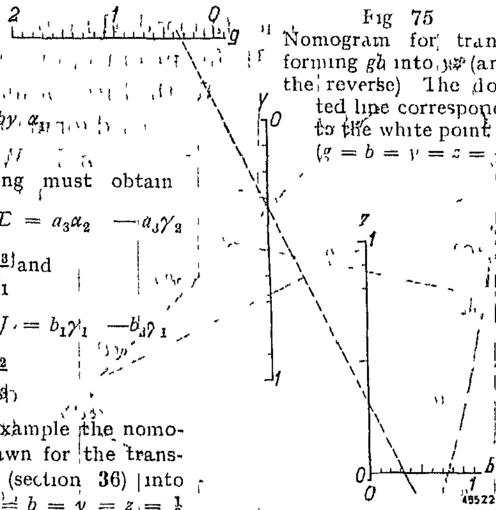


Fig 75 Nomogram for transforming gb into yz (and the reverse) The dotted line corresponds to the white point E ($g = b = y = z = \frac{1}{3}$)

§ 61 *Other subjective colorimeters*

For the colorimeters so far discussed we had three fixed colours at our disposal, and the matching of the colour to be examined was brought about by additive mixing. We shall now discuss some colorimeters based on slightly different principles.

Closely related to the trichromatic colorimeters is R i c h t e r's instrument, already mentioned The only deviation is that this instrument has no less than 9 primary colours at its disposal These nine colours have been indicated in fig 76 One of the primary colours is the white point W ($x = y = z$, the German standard illuminant E, see section 23), the other colours consisting of the

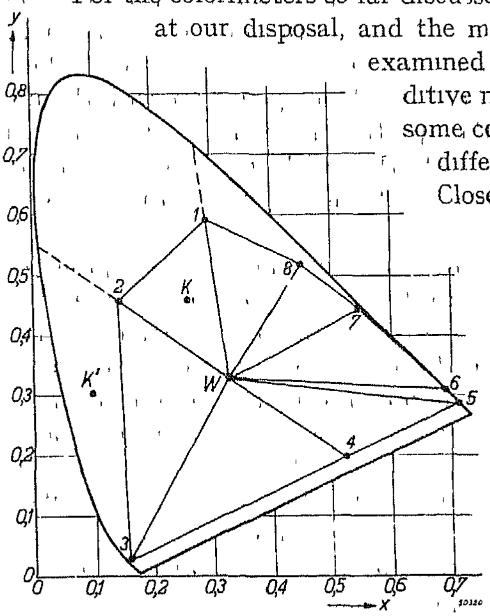


Fig 76
The primaries of R i c h t e r's colorimeter
The colour K is matched by mixing W, 1 and 2

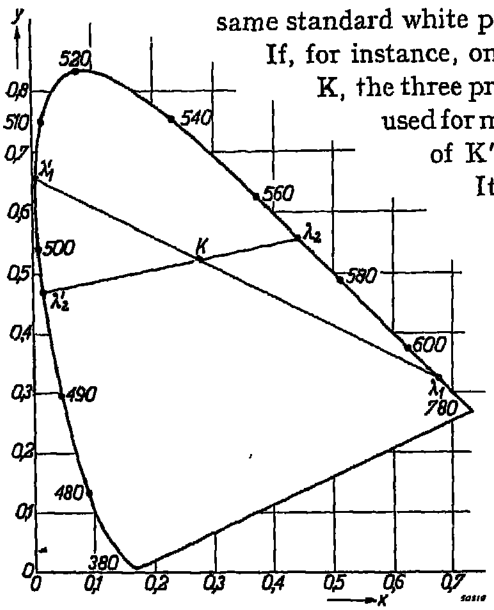


Fig. 77
Guild's vector method. K is matched successively by mixing λ_1 with λ_1' and λ_2 with λ_2'. The colour point is the point of intersection of $\lambda_1\lambda_1'$, and $\lambda_2\lambda_2'$.

same standard white provided with different filters. If, for instance, one wishes to measure a colour K, the three primary colours W, 1 and 2 are used for matching; for the measurement of K' W, 2 and 3 are used, etc. It is clear that negative quantities of W are only sometimes required; these are obtained by the "optical switch over" described in sect. 60. One can usually decide at first sight in which triangle (W12, W23 etc.) K lies. It is only necessary to consider the hue.

The advantage of such a subdivision of the range of colours lies in the fact that the position of K can be determined more accurately by matching with the aid of the primary colours W 12 than, for instance, by matching with the colours 1, 3 and 5, which are much farther from K. Another advantage is that with the use of W, 1 and 2 as a rule the spectral distribution of the match resembles that of K more closely than with the use of 1, 3 and 5. The result is that the individual differences between the observers is less significant.

Farther removed from the trichromatic colorimeters is the group of instruments whose purpose is to define the colour to be measured at once in the monochromatic system. In these instruments the matching therefore takes place by mixing a standard white with a spectral colour. Equality is again obtained by adjusting three magnitudes, i.e. the choice of the spectral colour (the dominant wavelength) and the quantities of white and of this wavelength required (the ratio of these quantities determines the colorimetric purity) The best known instruments of this type are those of Nutting [1]) and Priest [4]). As compared with the best trichromatic colorimeters they have the following disadvantages:

a. very strong divergence between the spectral composition of the two halves of the field of vision;

b. as a rule brightness equality is required between greatly differing colours (white and a spectral colour), which always leads to uncertainty and individual differences.

Beside these disadvantages the advantages fall well into the background. These are: a slightly easier adjustment, since λ_d can be adjusted fairly accurately at the beginning and so there are only two variables to regulate, and a saving of some arithmetic if we are only interested in λ_d and p. For these reasons this type of instrument has lately become quite obsolete. For their technical construction see the discussion in sect. 57-60.

A system of colour measurement invented by G u i l d [3]) is illustrated in *fig. 77*. In order to define the position K in the colour plane of the colour to be measured, the observer first finds two wavelengths λ_1, λ_1' capable of matching the given colour by additive mixture. Without having to determine the ratio of the two components λ_1 and λ_1', we know that K must lie on the straight line $\lambda_1\lambda_1'$.

If now a second set of wavelengths $\lambda_2\lambda_2'$ is determined, possessing the same property, then the position K is fixed at the intersection of the lines $\lambda_1\lambda_1'$ and $\lambda_2\lambda_2'$. Instead of λ_1 and λ_2 two fixed lights (either spectral or otherwise) may be taken and λ_1' and λ_2' determined in the same manner once more.

The advantages of this method are the avoidance of sources of error connected with the determination of the ratio of the components, the simple construction of the instrument with which the measurements can be effected, easy control of the results, namely by finding a third pair of wavelengths (the three lines must then pass through one point), and the avoidance of arithmetic.

Against this stands the often repeated disadvantage that the results depend to a very great degree on the individual properties of the observer, owing to the great difference between the spectral composition of the two lights to be compared. G u i l d himself calls this the "vector method".

A method described by R ö s c h [1]) is based on the property discussed in chapter VII that colour sensation can be matched by an ideal colour. The colour can therefore be characterized by the two wavelengths λ_1 and λ_2 occurring in the ideal colour, and its brightness. R ö s c h, whose method is specially intended for the measurement of coloured objects, has chosen three other quantities, namely the width of the interval in which the ideal colour radiates light, the "average" wavelength of this range and the ratio of the brightness of the colour measured to that which the matching ideal colour

would assume under the same illumination (in other words, the ratio of the reflection factor of the colour measured to the greatest value that it can assume with the given illuminant and in the given point of the colour plane).

Rösch has constructed an instrument with which it is possible to measure these three quantities directly. The ideal colours appear by projecting a spectrum of the illuminant and screening off a particular part of it. The advantages of this method are the simple construction of the instrument, the slight influence of the individual properties of the observer and fairly easy adjustment. The method has the disadvantage that it does not supply the trichromatic coordinates directly, but this can be minimized by using suitable tables and graphs.

Finally we must mention another group of instruments in which the colour is reproduced by subtractive mixing. We let the light of the — white — illuminant pass through a combination of filters placed one behind the other. As a rule these filters are chosen from a series of blue, yellow and red specimens, and at first the colour to be examined was defined by the numbers of these series used in the adjustment [Lovibond [1]].

The colour was in fact defined by three numbers but, as a result of the use of subtractive mixing, these three values are quite unsuitable for making simple calculations, as can be done with the trichromatic coordinates. It is therefore a first necessity for us to be able to calculate the XYZ coordinates from the values found.

The connection between these two groups of magnitudes is generally complicated and not clearly to be seen. For Lovibond's instrument, called a *tintometer*, it has been possible to simplify these calculations considerably [Schofield [1]), Fawcett [1])].

On account of its very simple construction this instrument has found its way into various fields in which it can be applied.

Instruments of this group have also been developed in which the colour can be continuously varied, for instance by the use of filters with varying thickness of the absorbent layer [Kallab [1]), Jones [2])].

CHAPTER X

Defective colour vision

§ 62 *The normal and abnormal eye. Survey of defective colour vision*

In chapters III-IX our attention was engaged exclusively with the ability of the normal eye to adjust two colour sensations until they are indistinguishable.

This sharply defined subject of research can now be extended in two different directions. In the first place we can still restrict ourselves to equality adjustments but this time take abnormal eyes into consideration as well. In the second place we can turn our attention to other powers of the organ of sight than those mentioned above. In chapter X we shall restrict ourselves mainly to the first extension of the subject and in chapters XII and XIII the second extension will be discussed. The first question that arises is: "When shall we call the eye of an observer normal?"

It is difficult to give a definite answer to this. In any case we shall have to limit ourselves to the properties of the eye which govern colour sensations in deciding what is normal or not and particularly the power to judge equality of colour. A short-sighted eye can be absolutely normal as regards the observation of colour.

One might, of course, put it thus: an eye is to be called normal if it answers to the "average" properties of the eye established by the C.I.E. system but in this case we should seek in vain for a normal eye, for such an intricate living organ as the eye always shows individual deviations. We shall therefore have to call an eye normal if its properties deviate only slightly from the established averages. What exactly is meant by "only slightly" is left to some extent to our discretion. The same uncertainty attaches to the idea of "average height".

Yet there is a real distinction between the concepts of "average height" and "normal sight", namely that: if we count in succession the number of men with the heights of 5′8″—9″, 5′9″—10″, 5′10″—11″, 5′11″—6′, 6′—6′1″.... 6′6″—7″, 6′7″—8″...., we shall see that the number of men (n) belonging to each of these groups continuously and regularly increases and then decreases according to a

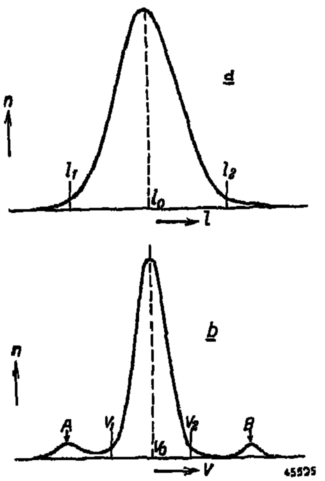


Fig. 78
Normal and abnormal body length and colour vision. a) Frequency of different body lengths. b) Frequency of mixing proportions v obtained with a certain colour-ajustment.

particular law (probability curve). *Fig. 78a* shows a diagram of this curve. We shall now call the men whose height lies between say l_1 and l_2 of "average" height. If the same calculation is made for certain properties of colour vision — for instance the ratio v in which the observer has to mix a certain spectral red and green in order to match the yellow of the sodium lamp — quite a different phenomenon is often found. For relatively small deviations from the most frequent value v_0, a similar curve is found to that of the height measurements *(fig. 78b)*, but with increasing deviations the number of persons often decreases less steeply than might be expected. The curve becomes practically horizontal and may even show a new maximum (A and B in fig. 78b) and sometimes it happens that between the primary maximum and a secondary one B there is a space where $n = 0$. Not a single observer has mixture ratios in this region. The maxima at A and B point to the existence of a certain number of persons with a peculiar defect of colour vision. That this is actually a new phenomenon and not a coincidental divergence from the average can be confirmed in two ways. In the first place the observers belonging to the region indicated by B display various other peculiarities of colour vision. In the second place — and this is more important — these defects are *hereditary*. They recur in later generations according to fixed laws (see section 67). So while we cannot say exactly where the mixture ratio should lie, in order to be able to call an observer normal (for instance between v_1 and v_2) we can say with certainty that observers whose mixture ratios lie in the neighbourhood of A and B show a deviation from the normal.

Now there are a number of types of such deviations; we have already seen two in fig. 78b.

In order to obtain a provisional general survey of these types we must remember what role the number three plays in the colour vision of the normal observer (sect. 5 and 6). The set of colour sensations of the normal observer is three-dimensional. With some abnormal observers

this collection may be either three, two or one-dimensional. There is never a greater number than three.
Now the different forms of defects can be subdivided according to this characteristic (one, two or three-dimensional) and we then have the following general classification.

1. People whose set of colour sensations is three-dimensional: these can reproduce all colours by the mixture of three fixed colours. These persons, called trichromats, can be divided into the following groups:
   a. normal trichromats, or in short, normal observers;
   b. anomalous trichromats of the first kind, called protanomalous;
   c. anomalous trichromats of the second kind, called deuteranomalous [the existence of the two secondary maxima in fig. 75b is due to the existence of the two groups (b) and (c)].
   d. anomalous trichromats of the third kind, called tritanomalous (very rare).

2. People whose set of colour sensations is two-dimensional: these can match all colours by mixing only two fixed colours. These people, called dichromats, can be divided into the following groups:
   a. dichromats of the first kind, or protanopes (formerly called red-blind);
   b. dichromats of the second kind, or deuteranopes (formerly called green-blind);
   c. dichromats of the third kind, or tritanopes (formerly called blue blind or violet blind; very rare).

3. People whose set of colour sensations is one-dimensional: these cannot distinguish any colour whatsoever; to these any two given colour sensations can be made equal to each other simply by varying the brightness. These people, called achromats, monochromats or totally colour blind, can be divided into the following groups:
   a. monochromats with blind cones (rather rare);
   b. monochromats with colour blind cones (very rare).

We shall only apply the much used term "colour blindness" to monochromats, whose colour sight is really missing.
This classification is based exclusively on the mixture laws holding for different observers, therefore on adjustments to equality. Of the other properties of the eye, the way in which the dichromats see colour, for instance, is particularly important. Does a dichromat,

unable to distinguish between certain green and red papers, see them in the same way that a normal observer sees the green paper? Or does he get an impression from both papers corresponding to that which the normal observer receives from the red one? This question cannot be solved for ordinary dichromats; for the only means of communicating our subjective sensations of a colour to another person is a description by means of colour names, and the colour names given by the dichromat are not exclusively determined by his colour sensations but chiefly by his obstinate efforts to make his denominations agree with those of the normal observer. Very rare cases have been ascertained in which one eye of an observer is normal and the other dichromatic [Hippel[1]), Holmgren[1]), Dieter[1]), v. Trendelenburg,[5]) v. Kries[3])]. In these cases the question can be answered as the observer can compare the impressions of the one eye with those of the other.

All the defects of the eye mentioned are congenital, hereditary and practically unalterable during life. We shall not discuss here the far less frequent defects that may arise during life (by injuries, poisoning, disease). These defects are not hereditary and may change in the course of a lifetime, perhaps even disappear.

See Judd[15]) for a description of these defects.

§ 63 *Anomalous trichromats*

These deviate least from the normal. Although this type of defect occurs most frequently (60 to 70 % of the abnormal cases belong to this group), knowledge of their properties is still very incomplete and fairly confused. There are various reasons for this. In the first place the mutual differences within the groups of protanomalous and deuteranomalous persons is much greater than in the other groups. This can go so far that there are cases in which it is doubtful whether we have a protanomalous trichromat or a normal trichromat. (With deuteranomalous persons the distinction between normal and abnormal observers is much sharper.)

In the second place the differences between anomalous and normal trichromats in the results of equality adjustments are considerably smaller than the differences occurring in some other faculties of the eye. As an example the discrimination of wavelength differences may be mentioned. For the normal observer a certain minimum difference (limen) is required between the wavelengths of two spectral colours if he is to perceive a colour difference between the two lights. Now

this limen is as a rule considerably greater for the anomalous trichromat than for the normal trichromat; protanomalous and deuteranomalous persons are less sensitive to wavelength differences. The problem becomes still less manageable since the limen of deuteranomalous people, for instance, can vary very greatly and the deviations in limen need on no account run parallel with the deviations of the equality adjustments. There are observers with whom the results for the equality adjustments deviate very much from those of normal trichromats, although their limens are only slightly raised, and vice versa [Nelson [1]]. It is to be understood that it is difficult for people whose sensitivity to colour differences lies far below the normal to determine trichromatic coordinates with any accuracy.

The following rules hold for colour matching by anomalous trichromats:

a. The three laws of Grassmann (sect. 24-26) remain unchanged; the number three plays the same part as with normal observers.
b. The quantitive results of matching experiments — which led, for instance, to table B for normal trichromats — are different from those of normal observers.
c. The different classes into which the anomalous trichromats are divided (see survey in section 62) are also distinguished by different results in these experiments.

Hence it appears that for the complete study of anomalous trichromats it is necessary to determine the trichromatic coordinates of the spectral colours for each observer. This determination is very intricate and since it is desirable to extend this investigation to as many individuals as possible, and also because of the great individual spread among anomalous trichromats, substitutes have been sought for these measurements. One of these takes the shape of a spot test which quickly enables us to separate the normal from the anomalous trichromats, and at the same time to sub-divide the latter into protanomalous and deuteranomalous (Tritanomalous people are excluded on account of their extreme rarity.) Such a test consists of the matching of one colour sensation by mixing some chosen lights. It must be remembered in setting up the test that the spectral distribution in the two halves of the field of vision to be compared must differ considerably, for an adjustment to equality of colour sensation in which equality of spectral composition exists at the same time, is also judged correct by people with a very defective

sense of colour, and therefore forms no reliable test for our purpose. The test consists in this, that the person tested must match the colour of sodium light (589 mμ) by mixing red light (671 mμ or lately more usually 665 mμ) and green light (535 mμ or 537 mμ).

The mixture is slightly less saturated than sodium light but the difference is so slight (p = 0.983 — 0.986 instead of 1.000) that in general no inconvenience is experienced.

The normal observer requires a particular brightness ratio of the red and green lights for this match. The anomalous trichromat, however, requires a ratio generally differing from the normal, while at the same time the distinction between protanomaly and deuteranomaly comes clearly to the fore. The protanomalous require a smaller ratio green/red and the deuteranomalous a larger (therefore relatively less red) than do normal observers. If therefore in fig. 78b the "mixture ratio" represents this green/red ratio, then the maximum A is due to protanomalous and maximum B to deuteranomalous trichromats. According to Trendelenburg [4]) and Schmidt there is a region between B and v_2 where $n = 0$, while this is not the case between v_1 and A. In other words, this test completely separates the deuteranomalous from the normal people, while between the protanomalous and the normal there is still a very small number of transition cases. According to Nelson [1]), however, such boundary cases would also occur with deuteranomalous people. The test described here, which apparently answers its purpose very well, was first introduced by Rayleigh [1]). An instrument (anomaloscope) which is in frequent use has been constructed by Nagel [1]) for carrying out this test [see also L. Hardy [1]) and Shaxby [1])].

The considerable width of the chief maximum in fig. 78b is to be ascribed mainly to the more or less yellow colouring of the macula lutea (sect. 4). By replacing the wavelength of sodium light by other wavelengths we can find out in dubious cases whether the observer is an anomalous trichromat or one with a strong colouring of the yellow spot. [Schrödinger [4])].

If we determine the trichromatic coordinates of the spectral colours for anomalous trichromats, defective colour vision (relative to that of the normal observer) seems for protanomalous people to be restricted chiefly to the red curve (the $\overline{X}$ of the C.I.E. system); this curve is shifted to shorter wavelengths. Directly connected with this is a displacement of the relative sensitivity curve which shows itself particularly in a low sensitivity to orange and red colours: the protanomalous trichromat sees the spectrum shortened on the long wavelength side. In the case of deuteranomalous people the defect lies

chiefly in the green curve ($\overline{Y}$), which is here shifted to longer wavelengths, while the same holds for the relative sensitivity curve. The results of different investigators [Kohlrausch[3]), Schmidt[2]), Nelson[1])] are very divergent.

Actually one should understand by the red and green curves mentioned above the trichromatic coordinate curves relative to the three axes selected in the isochrome directions of the dichromats (sect. 64 and 65).

The practical importance of the detection of anomalous trichromats lies in the fact that they are less suitable for some occupations. Their defective colour judgment will only be an impediment in very special occupations (for instance they are unsuitable as observers in colorimetry by subjective methods). They will, however, be much more impeded by defects in other properties of their vision, for instance, their lack of sensitivity to colour differences. Particularly when required to distinguish and recognize colours under difficult circumstances their achievements will usually lie far below those of normal trichromats [Pitt[2])].

§ 64 *Dichromats*

Dichromats show a much more defective colour vision than the previous group. While most anomalous trichromats are unaware that they have defective vision, dichromats know very well that their colour sight is abnormal.

The most important properties of dichromats are the following:

a. Grassmann's laws still hold, but with this difference, that in the first law (section 24) the number three must be replaced by the number two; only two primary colours need be chosen; the set of colour sensations is two-dimensional for dichromats.

b. When a normal trichromat has made the two halves of the field of vision equal by matching a colour from three primary colours, the two halves will still appear equal to the dichromat; the dichromat will agree with any adjustment made by the trichromat (this is not the case with the anomalous trichromat!). This fact was ascertained by Seebeck[1]) in 1837.

The protanope generally also agrees with colour adjustments made by a protanomalous observer, the deuteranope with those of a deuteranomalous observer. This proves the mutual relationship between the two protodefectives and between the two deuter-defectives. We find this relationship confirmed when studying heredity (section 67).

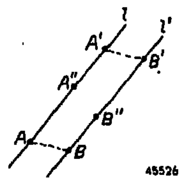


Fig. 79
Isochrome directions for a certain type of dichromats. When for a dichromat A and A′ are indistinguishable the colours on a line (l′) parallel to (l) are also indistinguishable.

c. The inverse reasoning, however, does not hold: when a dichromat has made an adjustment to equality the trichromat will generally disagree with it. This was to be expected: since the set of colour sensations of the dichromat is so much poorer than that of the trichromat it must repeatedly happen that the dichromat calls two colours equal which differ considerably to the trichromat. It is not a matter of very small differences, which, on account of a lower sensitivity to colour differences, would remain unobserved, but often of enormous differences such as a red and green colour which the dichromat calls equal, or a particular colour of the spectrum which the trichromat calls blue-green while the dichromat sees no difference between that colour and white (daylight)! The dichromat therefore can not distinguish certain colour differences existing for the trichromat.

d. The differences between the three types of dichromat lie in the fact that different colour confusions are made by the protanope, the deuteranope and the tritanope.

How does the dichromat react to the XYZ space and the colours arranged therein by the normal trichromat? All colours appearing the same to the trichromat and which he places at one point in the space are also indistinguishable to the dichromat [property (b)]. The dichromat will therefore approve of the arrangement made in so far that for him too a particular colour sensation corresponds to each point of the XYZ space. But he will disapprove of the fact that there are frequently two different points in the colour space whose colours made the same impression on him [property (c)]! We shall now trace how these points are distributed in the colour space. For the present we shall keep to one type of dichromat and suppose that the colours that the normal observer places at point A (coordinates XYZ) make the same impression on him as the colours placed at A′ (coordinates $X'Y'Z'$) *(see fig. 79)*. We can then show, in the first place, that all colours ranged on the straight line AA′ invoke the same colour sensation in him.

Any given point A″ lying on that straight line will have the coordinates $kX + (1-k)X'$; $kY + (1-k)Y'$ and $kZ + (1-k)Z'$. For the colour A″

both for the trichromat and for the dichromat [property (b)] the following will therefore obtain:

$A'' \longleftrightarrow kA + (1-k)A'$

But for the dichromat the following also holds:

$A' \longleftrightarrow A$ or $(1-k)\ A \longleftrightarrow (1-k)A'$

(Grassmann's third law) and therefore:

$A'' \longleftrightarrow kA + (1-k)A \longleftrightarrow A$

(Grassmann's second law).

In the second place we can prove that all points on a given line l′ parallel to l contain colours which are again indistinguishable to the dichromat.

Let B (coordinates $X + \Delta X,\ Y + \Delta Y,\ Z + \Delta Z$) be a point of l′; complete the parallelogram AA′B′B then B′ has the coordinates $X' + \Delta X,\ Y' + \Delta Y,\ Z' + \Delta Z$. But now as A (coord. XYZ) $\longleftrightarrow$ A′ (coord. $X'Y'Z'$) also B $\longleftrightarrow$ B′ (Grassmann's second law).
For any given point B″ of l′ the proof is analogous to that of A″.

In summarizing we can say, therefore, that the XYZ space contains a set of parallel lines with the property that all colours placed on one particular line by the normal trichromat are indistinguishable to the dichromat. Otherwise expressed, each line of that system forms a straight line of constant colour for the dichromat. As we call lines of constant temperature isotherms, so we can indicate these lines of constant colour by the name of isochromes.
For a dichromat belonging to one of the two other types we can also indicate a system of isochromes but their common direction is different according to whether we are concerned with a protanope, a deuteranope or a tritanope.

The isochromes that can be drawn through the origin of the XYZ coordinate system are generally called "Fehlfarben (missing colours) der Dichromaten" in German. We have given no literal translation for this as in the first place these "Fehlfarben" are not colours (the straight lines referred to lie outside the colour cone), and in the second place the addition "Fehl" (= missing) recalls too strongly a certain theory the correctness of which has yet to be proved.

§ 65 *Dichromats (continued)*

As described in section 64, three directions therefore appear — the isochrome directions — with the inherent property that protanopes observe no colour change along the first direction, deuteranopes no colour change in the second direction and tritanopes no change in the third direction.
We can make good use of the existence of these directions by introducing a new set of coordinates suitable for defining the colour

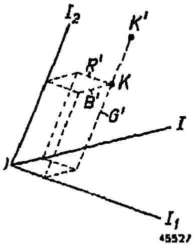


Fig. 80

Isochrome directions OI_1, OI_2, OI_3 for the protanopes, the deuteranopes and the tritanopes respectively. For the fixation of the colour point of a dichromat twq of the three coordinates R' G' B' are sufficient.

sensations of both trichromats and dichromats in a simple way.

Let K be a point in the XYZ space at which the trichromat has defined a colour; for this definition we always need a trichromat, while the dichromat must always accept that decision, see property (b). Further let O be the origin of the XYZ coordinate system, and OI_1, OI_2 and OI_3 the lines that can be drawn through O in the three isochrome directions. These directions do not, of course, coincide with the X, Y and Z axis nor with the O1, O2 and O3 directions we used in chapter IV as axes (fig. 10b, fig. 12); but we *can* use them to characterize the point K already otherwise defined by three points in the space, namely by the distances from K to the three planes OI_1I_2, OI_2I_3 and OI_3I_1. These distances — which must of course be measured again in the directions of the axes, in this instance in the isochrome directions — are indicated in *fig. 80* by R', G' and B'.

The accents are intended as a distinction from the *RGB* system of the C.I.E. mentioned in section 36

The advantage of the coordinates $R'G'B'$ over XYZ is that the values $R'G'B'$ of course completely define the colour K for the trichromat, while in order to define the colour sensation of a dichromat we need only know two of the three quantities. If we take as an example the deuteranope who has OI_2 as his isochrome direction, we obtain point K' by changing G' (leaving R' and B' unchanged). But the colour defined at K' for the trichromat creates the same sensation in the deuteranope as that of K (for KK' is for him the isochrome direction). The quantity G' has therefore no influence on the colour sensation for the deuteranope, so that we need only know R' and B' in order to define his colour sensation. In exactly the same way we can prove that the colour sensations of the protanope can be entirely established by G' and B' and those of the tritanope by R' and G' So we have indeed found in the $R'G'B'$ coordinates a system suitable both for trichromats and for dichromats, while the property that two quantities are only needed for the dichromat illustrates in a striking manner the fact that for them the set of colour sensations is only two-dimensional.

We have illustrated above the position of the dichromat in relation

to the colour space of the trichromat; now we wish to ascertain his position as regards the colour triangle derived from this.

For this purpose we make use of the geometrical connection between a colour space and the colour triangle derived from it (see close of section 27 and fig. 21).

In *fig. 81* we have shown the XYZ space with a plane V cutting off equal parts of OX, OY and OZ: the colour triangle originates in this plane. Further one of the isochrome directions has been drawn through O, which cuts plane V at point A (the isochrome direction is therefore chosen in the OXY plane; we shall see later on that two of those directions do indeed lie in this plane).

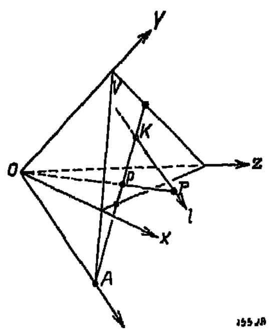


Fig. 81
Colour space and colour plane for a dichromat, V = colour plane, OA and ( ) isochromous directions. All colours on KA display the same chromaticity for the dichromat.

Finally we consider a colour point K in V and draw the line l through K parallel to OA. Now the dichromat selected sees all colours on line l exactly as K. In order now to find the points in the *colour triangle* whose colours make the same sensation on the dichromat as K, we join K to A. As l is drawn parallel to OA the straight lines OA, l and AK all lie in one plane. According to section 27 we now find the point in the colour triangle corresponding to a point P of l at the intersection of the line OP with V. But OP also lies in the plane of OA, l and AK, therefore the point p sought for in the colour triangle lies on the line of intersection AK of the said plane with V. We therefore see that all colours which, for the dichromat, correspond with K in the colour space lie on l and in the colour plane on the line AK.

In *fig. 82* the colour triangle has been drawn separately with the point of intersection A (fig. 81) and some colour points; the lines AK, AB and AK' therefore contain the colours which for the dichromat correspond with K, B and K' respectively. We now see also that there is a monochromatic light — with wavelength λ_0 — that makes the same impression on the dichromat

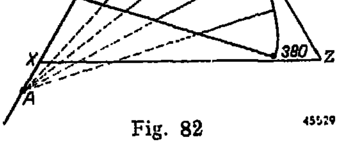


Fig. 82
Colour plane for a dichromat. All colours on a straight line through A appear to him with the same chromaticity. The spectral colour λ_0 (neutral point) is for him equal to white (B).

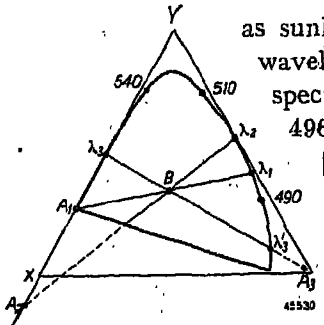


Fig. 83
Colour plane for the three types of dichromats. A_1 A_2 A_3 are the intersections of the colour plane with the isochrome directions through O (see fig. 80) λ_1, λ_2, λ_3 and λ_3' are the neutral points of protanopes, deuteranopes and tritanopes respectively.

as sunlight (standard illuminant B). Light of wavelength λ_0 we call the neutral point of the spectrum. For protanopes this point lies at 496 $m\mu$, for deuteranopes at 500 $m\mu$ [Pitt[1])]; in the rare cases of tritanopy we see a neutral point occurring at about 570 $m\mu$ and according to some observers a second neutral point in the blue. That this second neutral point was to be expected appears from *fig. 83* in which the points of intersection $A_1A_2A_3$ of the axes OI_1, OI_2 and OI_3 given in fig. 80 have been drawn with the colour plane as calculated by Bouma[10]) from the measurements of Pitt. For protanopes λ_1 is the neutral point, for deuteranopes λ_2 and for tritanopes we see the two points λ_3, λ_3'.

If we draw a number of straight lines through each of the points A_1, A_2 and A_3 (the same as those through A in fig. 82) we can read off directly what colours are confused by the various dichromats. Thus both the protanope and the deuteranope confuse the red, yellow and greenish yellow colours of the spectrum.

The positions of the points A_1, A_2 and A_3 are not accurately known. The early measurements of König[1, 6]) were carried out with far from perfect instruments, while the publication of newer measurements by Pitt is in such a form that it is very difficult to determine the points accurately. In the latter case we are obliged to make use of the relative sensitivity curve of the protanope and the deuteranope, a datum which is much less reliable than the results of mere adjustments to equality.

The uncertainties have been an added inducement not to use the system *(R'G'B')* of fig. 80 for practical purposes, but the *XYZ* system. [See Bouma[11]) and A. A. Kruithof[8])].

The following additional theoretical speculations have attended the introduction of the coordinates R', G' and B' (fig. 80). Suppose three photochemical processes take place in the normal eye (cf. section 53) and that the coordinate R' is a measure of the stimulation of the first process, while G' and B' play the same part for the second and third processes: then the occurrence of dichromats would be simple to explain, for the deuteranopic observer does not notice if we only vary G', which leads to the obvious supposition that for deuteranopes the second photo-chemical process is inactive. In the same way we should be able to explain the properties of protanopes by the fact

that the first process is missing, and those of the tritanopes by the lack of the third process. This line of thought, originating from H e l m h o l t z and his collaborators, gives us a very attractive possibility for explanation, but it has so far been impossible to prove the correctness of the hypotheses. The same holds for other theories submitted for the explanation of the deficiencies discussed.

If we try the experiment described in section 63 on protanopes and deuteranopes we see that they only require two colours for an adjustment to equality. They can make the red and green indistinguishable to themselves merely by varying the brightness. Yet this test can serve to decide quickly whether the person is protanopic or deuteranopic, for in order to make the colour indistinguishable from the same green colour the protanope requires much more red light than the deuteranope (about 5 ×). This fact appears from *fig. 84* in which the various sensitivity curves are given: N for the normal observer, P for the protanope, D for the deuteranope [according to P i t t [1])].

* Renewed interest in H e l m h o l t z's theory has been aroused by the work of Hl. d e V r i e s [2, 3, 4]), who devised new methods of determining the response curves of the receptors. According to De V r i e s the red and green curves are identical with the luminosity curves of deuteranopes and protanopes respectively, whilst the blue curve practically coincides with he function $\bar{z}(\lambda)$ of the C.I.E. (It may be mentioned that this idea has already been put forward by K ö n i g.) The green curve of the deuteranomalous coincides with the red curve of the protanomalous, in accordance with a conjecture made by S c h o u t e n [2]). This curve is intermediate between the normal red and green curves and very near to the normal red curve. The fact that protanopes agree with the adjustments of the protanomalous and the deuteranopes with those of the deuteranomalous is explicable in this manner.

§ 66 *Monochromats or achromats; total colour blindness*

While we are still ignorant of the nature of the defects in the organs of vision which lead to the existence of dichromats and anomalous trichromats (hence the large number of different theories!) this is not the case with the defects leading to total colour blindness.

In the first and most frequent type of monochromats the defect consists simply of the

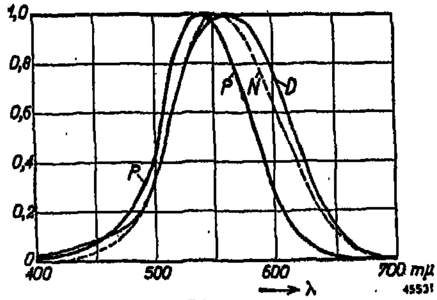


Fig. 84
Relative luminosity curve for the normal trichromatic observer (N) the protanope (P) and the deuteranope (D). Measurements of P i t t.

fact that the cones do not function at all and are therefore truly blind. The monochromat of this type can see only with the rods. This fact has a number of consequences which might have been predicted directly from the properties of the rods (sec. 4 and 12). The most important are the following:

1. The monochromat is unable to distinguish the various colours. He sees his whole environment in black, grey and white hues. The only difference that he can observe is the difference in brightness. If he is shown any two lights he can make the two sensations indistinguishable simply by varying the brightness of either of the two. His set of "colour" sensations is therefore one-dimensional.
2. In the central part of his retina there is usually a region that cannot give rise to a sensation of light (fovea centralis). In order to observe small objects he must always look slightly to one side.
3. At very low brightnesses the monochromat sees as well as the normal observer, but at high brightness he can see practically nothing: he is "light-shy", or photophobic.
4. His relative sensitivity is that of rod sight (fig. 3a).
5. His visual acuity is very low.

The cones are not entirely absent but their shape is different from those in normal eyes. Perhaps in some cases they have been partly replaced by rods.

In the second, much rarer type of total colour blindness the phenomena mentioned under 2-5 do not occur at all. All properties point to the fact that in this case the cones are normally present, maybe even the three photochemical processes are functioning. But by some defect or other the sensations of colour difference do not reach the brain of the observer. At what point a link is missing from the chain of processes is unknown.

Another theory is that the photochemical processes are all identical. Just as for the dichromats two processes overlap, so here do all three. Against this theory, which would explain monochromatism in the same way as dichromatism, the shape of the sensitivity curve bears witness. The sensitivity curve of this second type of total colour blindness agrees fairly well with the normal cone sensitivity (fig. 84 N).

The phenomena occurring with monochromats are so conspicuous that no special experiments are necessary to distinguish them from one another.

In summing up we can state that anomalous trichromats are unsuited for work requiring the accurate and rapid distinguishing of colours, dichromats are unsuited for all work in which colours play a

part and monochromats (with cone blindness) are deficient in every aspect of vision.

§ 67 *Heredity of the defects*

All the defects discussed in sect. 63-66 are hereditary. In studying the problem of heredity it strikes us at once that the manner of inheritance is not the same for all defects. This fact makes defective vision particularly suited to illustrate the general laws of heredity in mankind. We shall therefore go a little deeper into this interesting problem, although it lies partly outside the scope of this book.

The transmission of characteristics to posterity takes place via the germ-cells (ovum and sperm). In *fig. 85* the contents of these germ-cells are shown very schematically, in so far it concerns our subject. The germ-cell contains a number of particles called chromosomes (48 in human beings). The majority of these are arranged in pairs and occur in the same manner in the male and female germ-cell. An exception is formed by the so-called x chromosomes: of these two are found in the female germ-cell and only one in the male cell. This single chromosome is coupled with a chromosome possessing quite different properties, designated the y chromosome. The x and y chromosomes are also called the sex chromosomes. Before impregnation of the ovum by the spermatozoon takes place (fertilization) the cells have split in two (reduction division) as shown by the dotted lines in fig. 85. When fusion occurs two of the "halves" originating in this way unite to form a new cell which again contains the original number of chromosomes and can now develop into a new organism. If this new cell contains a y chromosome (if the parts A and D or B and D have united) a son is begotten, and in the other case (from the combination of A and C or B and C) a daughter.

Each hereditary characteristic is connected with a particular pair of chromosomes. At fertilization one of these chromosomes transmits the property or the defect to posterity. Now there are certain characteristics which are transmitted by the x chromosome. We say that these characteristics display "sex-linked" heredity.

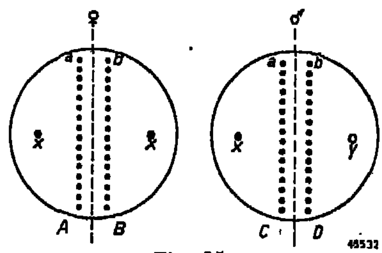


Fig. 85
Chromosomes: ♀ in the female germ cell, ♂ in the male germ cell. x and y sex chromosomes. In fertilisation one of the halves A and B combines with one of the halves C and D.

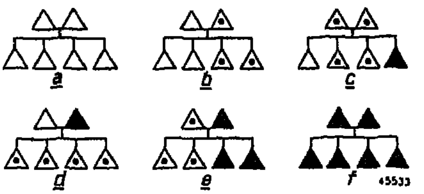


Fig. 86

Heredity of a recessive characteristic (not sex-linked). Empty triangles: normal persons. Black triangles: the characteristic shows itself. Triangles with dot: persons who may transmit the characteristic but who are outwardly normal.

Other characteristics are connected with other particular pairs of chromosomes [for instance (a) and (b)]; these characteristics are therefore not sex-linked. In the inheritance of the first group of properties the male and female germ-cells play different parts, but in the second group these roles are identical.

After fertilization two chromosomes form a new pair. It may happen that both chromosomes are bearers of the same defect or that only one of the two is defective. In the first case the son or daughter will also show this defect. The second case is more complicated and we must here distinguish between sex-linked defects and others. Among the non-sex-linked defects we find some which appear if only one of the chromosomes is the bearer of the defect. We then speak of dominant defects (the chromosome with the defective property dominates over the other). In other cases the defects do not appear in a child if only one of the chromosomes is the bearer. In this case we speak of recessive defects. The possession of one abnormal chromosome does not then lead to the appearance of the defect in the child, but it is present latently to the extent that it can be transmitted to the grandchildren in whom it may either appear or remain latent.

For the defects connected with the x chromosome we must distinguish between sons and daughters. If the daughter inherits one abnormal and one normal x chromosome some defect will appear (dominant characteristics) and others remain latent (recessive characteristics) In the son (who has only one x chromosome) the defect only appears if that particular chromosome is abnormal.

An example of a recessive characteristic which is not sex-linked is total colour blindness (cone blindness), discussed in section 66. (Other examples are: hereditary forms of epilepsy, mental deficiency and blindness, blue eyes, fair hair etc.). The way in which these characteristics — or defects — are handed down to posterity can be traced completely from what has been said above and is illustrated again in *fig. 86*. Here a black triangle represents a person in whom a particular characteristic appears, a triangle with a dot an outwardly normal person with, however, one abnormal chromosome and

finally a blank triangle an entirely normal person. Thus fig. 86c represents the following case: father and mother are outwardly normal but can both transmit the defect. They both therefore have one normal and one defective chromosome. In the children four cases can occur. In the first place the two normal chromosomes may have coupled during fertilization and then the child is entirely normal. In the second and third places the abnormal paternal chromosome and the normal maternal one may have taken part in the fertilization, or vice versa: hence the child will have one abnormal chromosome and is therefore outwardly normal but carries the defect latent with him. In the fourth case the child may have inherited two abnormal chromosomes, and in this case the defect will appear. These four cases occur with equal frequency, so that chance plays a decisive part.

From fig. 86 we can infer the following details:

1. Entirely normal parents have entirely normal children (a): if the same defect appears in both parents it also occurs in all the children (f).
2. Two outwardly normal parents may have a child in which the defect appears (c).
3. If one of the parents is entirely normal the defect cannot appear in the children (b and d).

It is the last point in particular that makes total colour blindness one of the very rare cases. When the defect is not very common, then after case (a), marriages (b) and (d) are far and away the most usual and in both cases the defect remains latent in the children.

An example of a dominant defect which is not sex-linked is the hereditary form of night blindness. This forms a parallel to total colour blindness. In this case the cones are normal but the rods do not function (other examples of dominant characteristics are brown eyes, brachydactyly etc.). *Fig. 87* shows the possibilities arising in the marriage of night blind persons. A blank triangle again represents a normal person, a half black triangle a night blind person with

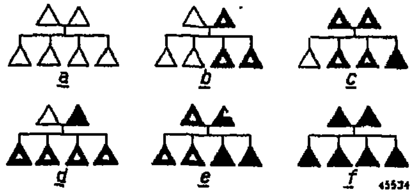


Fig. 87

Heredity of a dominant characteristic (not sex-linked). Empty triangles: normal persons; filled triangles: persons with two abnormal chromosomes; the characteristic shows. Partly filled triangles: one abnormal chromosome: the characteristic shows likewise.

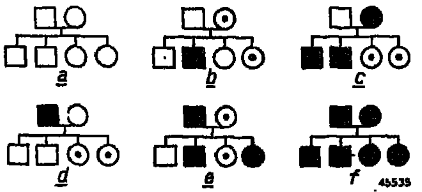


Fig. 88
Heredity of a recessive characteristic (sex-linked). Empty squares: normal men, empty circles: normal women; black squares: abnormal men, black circles: abnormal women. Circles with dot: women who can transmit the characteristic but are outwardly normal.

only one abnormal chromosome and an entirely black triangle a night blind person with two abnormal chromosomes. Fig. 87 differs from fig. 86 only in that the person with the latent defect is replaced by a night-blind person with only one abnormal chromosome. This is indeed the difference between recessive and dominant defects. Cases 1-3 are now as follows:

1. Normal parents have normal children (a).
2. If one of the defective parents has two abnormal chromosomes no normal children can occur (d, e, f). If this is the case with both parents (f) even normal grandchildren are impossible!
3. In special cases two night-blind parents can have a normal child (c).

Dominant defects often occur frequently in one family. Thus C u n i e r and N e t t l e s h i p [D u k e - E l d e r [1])] found in a pedigree extending over three centuries and containing more than 2,000 persons not less than 153 night-blind people! This is due to the fact that — after (a) — the most frequent unions, *viz.* (b) and (d), produce on an average 75 % of night-blind children!
Examples of recessive sex-linked defects are the various cases of dichromasy and anomalous trichromasy (another example: haemophily). *Fig. 88* shows the results of marriages of deuteranopes. Here we must consider the sex: men are represented by squares, women by circles. The black figures are dichromats, the circles with dots outwardly normal women able to transmit the defect. Finally, the blank figures are entirely normal persons. The following facts can be inferred from this diagram:

1. Completely normal parents have completely normal children (a); two deuteranopes have exclusively deuteranopic children (f).
2. Two outwardly normal parents may have a deuteranopic son but no deuteranopic daughter (b).
3. If the mother is normal the defect can appear in none of the children (d).
4. The condition of the sons is quite independent of that of the fathers [(a) and (d), (b) and (e), (c) and (f)], for the son inherits no

x chromosome from the father and the y chromosome cannot be the bearer of the defect.

5. Deuteranopy is much rarer among women than among men.

For, after (a), the most frequent marriages (b), (c) and (d) produce deuteranopic sons but no deuteranopic daughters.

The same diagram and the same conclusions also hold for protanopes, protanomalous and deuteranomalous persons.

Interesting cases occur when the father and mother have different defects. In such cases the daughter may possess two x chromosomes which are abnormal in different respects. If one defect is protanomaly or protanopy and the other deuteranomaly or deuteranopy no defect will appear but she may transmit either of them. If one defect is protanomaly and the other protanopy she is protanomalous; in the same way with the two deuter defects. Such cases teach us a great deal about the interconnection between the various forms of defects.

* Hl. de Vries[4, 5]), starting from the original Young-Helmholtz supposition that three different kinds of receptors exist in the retina, has found a method of determining the ratios of the "numbers" of these receptors, in particular the ratio of the number n_r of the red-sensitive to the sum $n_g + n_r$ of the numbers of green-sensitive and red-sensitive receptors, which we shall denote by q_{obs}.

$$q_{obs} = \frac{n_r}{n_g + n_r}$$

For protanopes $q_{obs} = 0$ and for deuteranopes $q_{obs} = 1$. De Vries supposes that the value q_{obs} is composed of the separate values of certain quantities q which are to be attributed to the different x-chromosomes. A female individual with chromosomes x_1, x_2 having the values q_1, q_2 will show on examination the value

$$q_{obs} = \tfrac{1}{2}(q_1 + q_2),$$

whereas for a male individual with chromosomes x_3,y, simply $q_{obs} = q_3$. According to the same author the q values of the separate chromosomes are hereditary. This leads to the following scheme of possibilities for the q values in a family, where a, b, c, denote numerical values different from zero.

father q_3	*mother* q_1q_2	*sons* q_3	*daughters* q_1q_2
c	ab	a or b	ac or bc
c	$0b$	0 or b	$0c$ or bc
c	00	0	$0c$
0	ab	a or b	$0a$ or $0b$
0	$0b$	0 or b	00 or $0b$
0	00	0	00

The relations between the values of q_{obs} of the members of the family which follow from this table have been confirmed experimentally.

Comparing this table with fig. 88 and remembering that in a male individual a q value zero results in $q_{obs} = 0$ but that in a female individual q_{obs} is only zero if both q are zero, we now understand better how it is possible that a woman may transmit the characteristic $q_{obs} = 0$ without showing this characteristic herself and why the same is not the case with a man. The measurements of De Vries therefore provide a "quantitative" insight into the phenomenon of recessivity.

TABLE 7	% observed		calculated
	men	women	women
Protanomals . . .	0·68	0·05	0·029
Deuteranomals. .	4·01	0·27	0·304
Protanopes	1·08	0·02	0·012
Deuteranopes . .	1·78	0·02	0·032
Totals.	7·55	0·38	0·38

From the laws of heredity we can calculate what relations there must be between the numbers of women and the numbers of men showing a particular sex-linked defect. Table 7 shows the percentage of the populations of Berlin suffering from the different defects [calculations by Schmidt[1])], and also the percentages for women which can be calculated from Schmidt's data for men. Considering the small number of abnormal women the correspondence can be called good.

Other calculations as to the frequency of the various defects can be found in Nelson[1]) (total of 8.82% in men), Waaler[1]) (8.01 in men, 0.44 in women), von Planta[1]) (7.95% in men); for the heredity of the defects see Just[1]), Waaler[1]), Brunner[1]), Trendelenburg[4, 6]), Kühn[2]), Gothlin[1]).

For a general resumé of Mendelian principles see Bateson[1]) or Gates[1]).

CHAPTER XI

The historical development of colour science

§ 68 *Introduction. The pre-Newtonian era*

To try to give a complete history of the development of ideas regarding colour vision would take us far beyond the scope of this work. For this would be a task in which an historian, a physicist, a painter, a physiologist and a psychologist would have to collaborate. We shall limit ourselves mainly to a few historical remarks on the subjects with which we have dealt in the previous chapters. We shall moreover take the opportunity of touching on a few subjects which have so far had no place in a more or less systematic arrangement; subjects which are partly only of historical interest but which may assist in deepening our insight into what has been discussed hitherto.

Where does the history of colour vision begin? Many are inclined to place this beginning as far into the past as possible. Some people place it at about the time of Homer, others in the Stone Age, while some even reach back to the era before Man had assumed human shape. Those who favour this last opinion are mostly engaged in considering the history of the development of the human organ of sight and of the power to distinguish colours [see Schrödinger [4])]. Undoubtedly such subjects are interesting but the theories constructed round them are so highly speculative that it would be as well not to attach too much value to them.

The material for such theories has mostly been derived from two different sources. In the first place a comparative study has been made of the structure of the eye (and — where possible — of the power to observe colours) of various species of animals. Among vertebrates eyes resembling the human eye in structure are usually found, but not all possess rods and cones. As a rule nocturnal animals and many water-dwellers only possess rods, while most rodents possess cones also (and sometimes exclusively). Probably only the last group of animals is able to distinguish colours. Scientists have succeeded in proving for various birds and mammals, by means of

training experiments, that their set of colour sensations is, as with human beings, three-dimensional. The anthropoid-apes seem to have a system of colour vision closely corresponding to that of man.

Treatises on the subject of colour vision among animals are to be found in works of de Voss[1]), Schnurmann[1]), Müller[1]), Gregg[1]), Kühn[1]), Bierens de Haan[1]), Trendelenburg[3]), Grether[1]), Birukow[1]), Rochat[2]), van Eck[1]), among others.

In the second place data on the history of development have been deduced from the occurrence of defective colour vision dealt with in chapter X and from other properties of the human eye.
Starting from all these data some authors have concluded that the human eye originated in a pure rod apparatus, that the first colour vision was dichromatic (split up into a blue and a yellow organ) and that the trichromats would only have made their appearance at a higher stage of development.
From coloured drawings found in caves and dating from the Stone Age we may conclude that in these times colour vision was already well developed. From what has been said about anthropoid-apes and the fact that even among the lowest types of the human race colour vision is practically the same as that of the West European, this might well have been expected. Any possible evolution in colour vision must therefore undoubtedly have taken place at a much earlier epoch. The fact that ancient Greek literature is poor in colour names certainly does not prove that (as some used to think) colour vision has appreciably developed in those few thousand years. In the Graeco-Roman civilisation period and later on at the time of the Renaissance (Leonardo da Vinci) literature about colours is found [Goethe[2]) collected much of this literature] but the ideas on which the present colour theory is based are by no means to be found there. These ideas are first met in the work of Newton, that is at the end of the seventeenth century. For us, therefore, the actual history of the science of colour begins in 1666 when Newton began his researches into this subject.

§ 69 *Isaac Newton (1642-1727)*

When Newton is mentioned one thinks first of the law of gravity named after him. In the second place comes his work on the subject of light and only thirdly does one think of the other important contributions which Newton made to all kinds of mathematical and physical subjects. This is an example of a typical phenomenon

which occurs repeatedly in the history to be studied here. Many of the figures which we shall encounter have become famous in quite different sciences. To prove the accuracy of this statement we need only add the following names: Da Vinci, Newton, Goethe, Schopenhauer, Thomas Young, Dalton, Maxwell, Rayleigh, Schrödinger. Truly a varied company! Perhaps the fact that the science of colours stands on the border of many other sciences has been the reason that so many men with many-sided interests have felt drawn to it.

We have already discussed in section 2 how Newton[1]) laid the first indispensable foundation for the accurate understanding of colour phenomena when he split up sunlight into its spectral components. The chapters in his well-known work "Opticks" dedicated to this subject form a real masterpiece in the field of experimental physics. Step by step we can follow the course of thought in Newton's mind: a course which led logically every time to the next experiment in answer to the questions raised by the previous one. And step by step we see how Newton in experimenting developed his instruments. We see him at one point close still further the "small hole in my window-shut" (through which the sunlight was allowed to enter) until it finally became a slit. We see him bring a lens to the beam of light and in this way, in principle at least, the first spectroscope came into being.

The spectra obtained in his first experiments were still very impure. As he did not then work with a narrow slit he observed the colours we saw appearing in fig. 46 (see sect. 45) as combinations of boundary colours. We can conclude from Newton's accurate data that the *value of $l : r$ in these experiments* amounted to about 0.27. We can therefore read from fig. 50 (see dotted line) what colours were seen by Newton in his first spectrum and by comparison with fig. 31 what the colorimetric purity was for the various colours. *Fig. 89* gives the results of such an investigation. We see that the colorimetric purity in this first primitive spectrum — for which neither slit nor lens was used — nowhere fell below 0.80 and that the colours for which the dominant wavelength is greater than 590 mμ appeared practically in absolute colorimetric purity.

Later on, when Newton saw that a purer spectrum was necessary, he improved his apparatus by introducing a narrow opening and imaging by lens, in such a way that $l : r$ was reduced to a value of 0.015, and then he could see all colours with a colorimetric purity of practically 1.

Besides the analysis of sunlight, the proof that spectral colours cannot be further resolved and the reconstruction of white light by additive mixing of all spectral colours, N e w t o n also gives

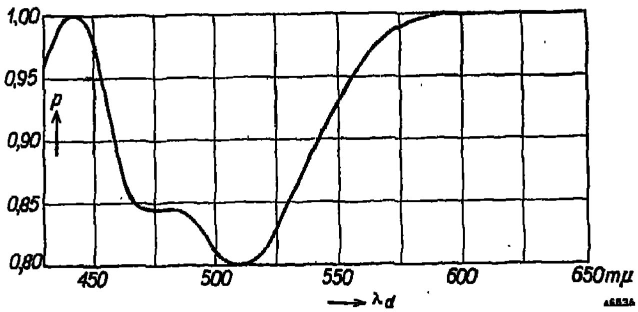


Fig. 89
N e w t o n's *first spectrum*; the purity p for different dominant wavelengths λ_d.

schematically the arrangement of the set of colours in the colour plane. He placed white in the centre of the diagram and the spectral colours round it, while the points within the circle represented the desaturated colours; the more desaturated the closer they lay to the centre. N e w t o n also realized that the use of a third dimension was necessary to distinguish the various colour sensations of the same chromaticity. He then arrived at the idea of representing a given colour by a small weight placed on the appropriate point of the colour plane, the size of which was equal to the intensity with which the colour appeared (see figure 2c). Then he made the surprising pronouncement that the position of a mixture of colours can be determined by calculating the *centre of gravity* of the set of weights representing the separate colours.

Was this pronouncement correct? As N e w t o n gave it, not quite, as he was not yet able to carry out accurate quantitive measurements. He had arbitrarily represented the curve of the spectral colours by a circle, and this choice was the reason why his statement was not quite correct, a fact which, remarkably enough, he mentioned himself.

But almost two centuries later when N e w t o n's circle had been replaced by the curve of the spectral colours as we now know it, and when what is understood here by intensity (or quantity of colour) had been established more accurately, it appeared that N e w-t o n's prophetic statement was correct in the strict sense of the word. In fact it could even be shown that this statement already

implied the Grassmann laws laid down 150 years after Newton!
In many books this "centre of gravity rule" of Newton is the corner stone on which the whole theory of colour is based.

For didactic reasons we have not done this. The difficulty lies in the concept of "quantity of colour" which Newton wisely did not define further. How this concept is to be understood appears from the following deduction of the rule from the ideas developed in the previous chapters (see sect. 33a).
If we mix a colour $(X_1 Y_1 Z_1)$ with a colour $(X_2 Y_2 Z_2)$ to a mixed colour (XYZ) then:

$$X = X_1 + X_2, \qquad Y = Y_1 + Y_2, \qquad Z = Z_1 + Z_2.$$

But now $x = X : (X + Y + Z)$ or $X = x\,(X + Y + Z)$ etc. If we substitute this in the above we obtain:

$$x\Sigma = x_1\Sigma_1 + x_2\Sigma_2, \quad y\Sigma = y_1\Sigma_1 + y_2\Sigma_2, \quad z\Sigma = z_1\Sigma_1 + z_2\Sigma_2, \tag{27}$$

in which $X + Y + Z = \Sigma$, $X_1 + Y_1 + Z_1 = \Sigma_1$, $X_2 + Y_2 + Z_2 = \Sigma_2$
If we characterize the colours (according to Newton) as $(x_1 y_1 m_1)$ $(x_2 y_2 m_2)$ and $(x\,y\,m)$, the centre of gravity rule demands:

$$xm = x_1 m_1 + x_2 m_2, \; ym = y_1 m_1 + y_2 m_2, \; zm = z_1 m_1 + z_2 m_2, \; m = m_1 + m_2 \tag{28}$$

The rule then only gives a correct method of calculating the trichromatic coefficients if the relations (28) are identical with (27). This is apparently the case if we make $m = \Sigma$; $m_1 = \Sigma_1$ and $m_2 = \Sigma_2$ (the last equation of (28) we find by adding the three equations (27)!
We find, therefore, that the rule is correct if we apply to each colour a mass equal to the sum of the trichromatic coordinates. In other words: by intensity or quantity of colour in this rule we must understand this sum. If we wish to deduce the whole colour theory from this rule we must introduce right at the beginning, besides the concepts of power and brightness, a third concept, namely that of the quantity of colour for the spectral colours. Such unnecessary complications in the fundamentals seemed undesirable.

In the few pages Newton devotes to the arrangement of colour sensations he sketches, as it were, the plan to be used in later ages for the construction of the whole edifice of the science of colour. If then we remember that Newton did not have a single predecessor on whose discoveries he could build further, we may draw the certain conclusion that even on this subject the master had no equal.

§ 70 *Johann Wolfgang von Goethe (1749-1832)*

In 1672 Newton published his first results; in 1704 the first edition of "Opticks" appeared. The whole of the eighteenth century in fact continued to build on the foundations laid by him.
Many copied Newton more or less literally, others added a little to what he said, but it cannot be said that there were any great

figures in this century in the science of colour. At the end of the eighteenth and the beginning of the nineteenth century a man already famous in quite a different field interested himself in the problem of colours. This man was *Goethe*[1]). It is almost impossible to compare two great men who differed so widely as Newton and Goethe and who had absolutely nothing in common with each other except that both studied the origin of colours. Was Goethe a great figure in this matter too? That is impossible to say. In any case he was not so important as Newton, but he was undoubtedly an extraordinarily interesting figure. Newton and Goethe started from totally different points of view, and, how could it be otherwise, approached the problem from quite different directions. This fact must be continually borne in mind if the different conclusions of both investigators and the almost hostile attitude of Goethe towards Newton are to be seen in the right light. Newton, an experimental physicist if there ever was one, took the view that quite independent of the observer there exists a great system, the nature and behaviour of which is regulated by strict laws, and he made it his purpose to discover some of those laws, a task in which, as we all know, he was extraordinarily successful. To study these laws one must set to work as objectively as possible and eliminate as far as possible the individual properties of the observer. This conviction is at once seen reflected in his first experiment. He admitted the light of the sun — the origin of everything — into his room through a small hole and let it pass through a prism so that a spectrum was produced on the opposite wall. He put the light to all kinds of tests, as it were, in order to discover something of its nature, and himself stood on one side as an interested but objective and unbiassed examiner.

How differently did Goethe face these problems! For him man was always the starting point; in this particular case the subjective colour sensations received and the numerous and divergent sensations they evoked in his fine artist's mind. The purpose of his investigations was to discover how these impressions originate and how they are influenced by circumstances. Typical of his point of view is his first experiment. He placed a prism before his own eye and observed through this first of all a white wall and after that a vertical division between a light and dark surface etc. In a sense he was not the examiner but the examined!

In continuance of his first experiments as the foundation stone of his theory, Newton discovered the *spectral colours*. Goethe on

the other hand discovered the *boundary colours* as we know them from section 44, as a "basic phenomenon". The answer to the question from which of these foundation stones the composite colours are actually built up, is this: they can, in principle, be considered either as mixtures of spectral colours or of boundary colours.

We can, for instance, imagine a light which contains only a narrow band of the spectrum, built up of two very slightly differing boundary colours. For a boundary colour R_1 (from 380 to λ) must be added to the said colour K (radiating between λ and $\lambda + \Delta\lambda$) to obtain another boundary colour R_2 (from 380 to $\lambda + \Delta\lambda$), or in the language of colour equations:

$$R_1 + K \longleftrightarrow R_2 \quad \text{or} \quad K \longleftrightarrow R_2 - R_1.$$

A colour showing an arbitrary continuous spectrum can be considered as the total of a large number of spectral colours but also as a combination of an equally large number of boundary colours. We can define the colour properties of an observer just as well by measuring the trichromatic coordinates of all spectral colours (those of the other colours then follow from Grassmann's law), as by taking the same measurements from a complete set of boundary colours.

In principle, therefore, Goethe and Newton were both equally right, but in practice Newton's proposition has such enormous advantages that Goethe's is entirely left out of the question and is never applied.

Some of these advantages are:

a. In building up from spectral colours we never have negative quantities to deal with (as was the case in the example $K \longleftrightarrow R_2 - R_1$).
b. A particular, commonly occurring class of colours, namely line spectra, can be much more easily broken up into spectral colours than into boundary colours.
c. Spectral colours play a much greater part in other branches of physics than the boundary colours.

Fig. 50, which we formerly considered as a transition from boundary colours to spectral colours, we can therefore now see as a continuous transition from the opinions of Goethe to those of Newton. In the first experiments of Newton there was actually still a transition stage (dotted line in fig. 50).

The reader will perhaps wonder why we bring out the contrast between Newton and Goethe. This is for historical reasons, for unfortunately, instead of supplementing the work of Newton from his own point of view, Goethe devoted the greater part of his considerations to the thankless task of disproving Newton's theories (in his own words "Enthüllung der Theorie Newton's"). We shall mention a few points in this conflict:

Newton states that sunlight consists of lights of various wavelengths (actually different refrangibilities) and that he separated

the rays in his experiments. Goethe on the other hand takes white light to be an indivisible unity containing no spectral colours. The colours would originate in the experiments through interplay with matter.

The last opinion is quite understandable when coming from such a man as Goethe. Many people will have wondered how it is possible for colourless daylight, which psychologically makes an impression of simplicity, to contain all those bright colours. Which of the two opponents is right depends mostly on individual philosophic taste. Most people will, however, prefer Newton's opinion, as it enables us to give a much simpler physical description of the phenomena.

We need not go deeply into the other point of difference, of whether the additive mixing of all parts of the sun's spectrum produces a white colour or a grey. In section 5 we saw that grey (just as brown) only occurs when there are other colours in the neighbourhood of inadequate brightness. This fact does not appear to have been clear either to Newton or to Goethe. Moreover this question was clouded by misunderstandings about the use of the word "white", which has so many meanings (see section 23).

A last difference between the opinions of Newton and Goethe concerns the colour "green". Goethe states (in agreement with the opinion maintained in artistic circles) that it is not a simple colour but can only be produced by a combination of yellow and blue. According to Newton there also exists a simple green, namely the spectral colour, while the blending of yellow and blue can produce white but never green. On this point, too, both investigators were right from their own point of view.

No green appears among the boundary colours, but it is produced by a combination of two boundary colours (sect. 45). And the divergent result of the blue-yellow mixing is explained by the fact that the combination of boundary colours spoken of in section 45 is actually a subtractive mixing (as in the mixing of pigments) while Newton had an additive mixing in mind.

§ 71 *Young, Grassmann, Maxwell etc.*

In this section we shall call to mind some historical facts which at first sight appear of minor importance but which have had an influence on the later development of the colour theory. The first name we must mention in this connection is that of *Thomas Young*

(1773-1829), another remarkable many-sided figure: a physician, physiologist, physicist, linguist, Egyptologist etc., he owes his greatest reputation to his work on the interference of light and his share in the deciphering of the Egyptian hieroglyphics. On the subject of colours he was the first to stress explicitly the significance of the number three (sects. 5-8). For he was the first to put forward the hypothesis that colour vision is based on the presence of three different light-sensitive organs [Y o u n g [1]], an idea taken up half a century afterwards and developed further by H e l m h o l t z and his collaborators. There are interesting details to be found about the life and work of Y o u n g in a series of articles that appeared in America on the occasion of the first centenary of his death [C r e w [1]), W i l l i a m s [1]), H e c h t [3])].

About 1800 the first development of views regarding the various kinds of defective colour vision occurred. Although dichromasy had undoubtedly been noticed centuries before, it appears to have first been explicitly mentioned in the literature by P r i e s t l e y (1777). Later on it became more generally known owing to the fact that the English chemist D a l t o n (1794) described his own defect in great detail. From this description we can infer that D a l t o n was a protanope. He again is a person, renowned in another realm of thought, who as a side issue interested himself with the problem of colours. The influence of that interest is still to be traced in the French collective noun "daltonisme" for the defective colour vision described in chapter X. Y o u n g, a contemporary of D a l t o n, already suggested that in such persons one of the basic processes did not function, another idea which H e l m h o l t z and his collaborators were to develop later on. At first no distinction was made between the two sharply divided cases of dichromats which to-day are indicated by the names of protanopes and deuteranopes. These two groups of abnormal observers were known as "red-green confusers". S e e b e c k [1]) (1837) was the first to ascertain explicitly the existence of the two different phenomena. Since then the names "red-blind" and "green-blind" were used, and these were only replaced by protanopes and deuteranopes in the twentieth century. The third class, the tritanopes (originally called blue-blind), were discovered and described much later [K ö n i g [10]) 1897].

That it took so long to discover that the distinction made by S e e-b e c k was correct was due to the methods by which defective colour vision was ascertained; methods which indeed enabled dichromats to be sharply distinguished from normal observers but to

which protanopes and deuteranopes reacted in about the same manner.
In the previous chapter we learned that the most suitable test for distinguishing various types of defective colour vision was that in which the observers were made to match a yellow colour by a mixture of red and green. Rayleigh [1]) (1882) was the originator of this test, who for this purpose matched the colour of the yellow sodium line (589 mμ) by an additive mixture of the red lithium line (671 mμ) and the green thallium line (535 mμ). Rayleigh's tests led to the discovery of anomalous trichromats and to the sharp distinction between protanomalous and deuteranomalous persons. In more recent times the colours of this "Rayleigh equation" were replaced by slightly different colours. For the investigation of tritanopic and tritanomalous people it appeared to be necessary to choose quite different colours [see for instance Engelking [1]), Trendelenburg [7])].
In the second half of the 19th century the further development of Newton's and Young's ideas took place, with which the names of Grassmann, Maxwell, Helmholtz and König must first be mentioned.
H. Grassmann (1809-1877) has the honour of enumerating explicitly and clearly the laws already implied in Newton's work, and thereby laying the foundation of the whole further development of the colour theory (1853). He too was a remarkably versatile man. Like Newton he devoted himself in the beginning to theological studies and later did very good work as a philologist, physicist and mathematician. He was also active in politics and music.
James Clerk Maxwell [1]) (1831-1879) was the first to measure and define the spectrum (1860). By this must be understood the experimental establishment of the trichromatic coordinates of the various spectral colours when they are shown in succession with the same power (distribution curves for the spectrum of equal energy). The measurement and definition of the spectrum leads, for instance, to a table like table B (see appendix). The Grassmann laws, together with the definition of the spectrum enable us to predict the result of each matching test. The weakest point of most spectrum determinations (particularly the older measurements) lies in the establishment of the power occurring in each of the spectral colours to be examined. Should we disregard this difficult part of the measurement entirely, we could only determine the trichromatic coefficients of the spectral colours, which would be inadequate for

calculating the same quantities for compound lights with known relative spectral distributions.

It would indeed be possible to calculate the trichromatic coefficient of the mixture from the luminosity ratio of the components (for instance with blended light), provided the luminosity coefficients of the system employed were known in some way or another (see calculation in section 37). More generally: we can calculate the distribution curves from the trichromatic coefficients if the relative sensitivity curves and the luminosity coefficients of the primary colours used for the determination are known. This less elegant method (undesirable mingling of colorimetry and heterochromatic photometry!) was also used in establishing the C.I.E. system [Guild [4])]:

Maxwell used as primary colours spectral lights of wavelengths 631, 529 and 457 mμ. For additive mixing he applied the method sketched in figure 69a (sect. 59). His measurements fell into oblivion later on, owing to the more complete and accurate measurements of König. The same fate befell the later measurements of Abney[1]).

§ 72 *Helmholtz, König etc.*

In the person of *Hermann Ludwig von Helmholtz* (1821-1894) we meet with a many-sided scientific talent and interest. Beginning his career as a physician, he made a name for himself as a physicist, mathematician, physiologist, philologist and musicologist, coupling all this with a talent for organization. It is largely owing to this last talent that he collected such a large number of excellent disciples and collaborators around him. In the domain which interests us here, his most important collaborator was undoubtedly *Arthur König*, a man with an extraordinary talent in the experimental field and who undertook research on countless subjects of a physiological-optical nature which have not been surpassed in the last thirty years.

Arthur König must be distinguished from two other people of the same name who were active in the same field at a later period, to wit, Albert König, the writer on Physiological Optics in the "Handbuch der Experimentalphysik" and Hans König, who in the last ten years has produced important work on the subject of photometry and colorimetry.

Helmholtz adopted Young's idea about the three fundamental processes, leaving, however, unexplained the nature of those processes, while Young explicitly speaks of three kinds of fibres in the optic nerve. The explanation of the existence of dichromatic persons suggested by Young was further developed by Helmholtz and König and supplemented by the idea that by investigating the properties of dichromats — especially of their iso-

chrome directions — it would be possible to arrive at a more quantitive formulation of the three-colour theory for normal vision; the spectral sensitivity of the three separate processes which, according to this theory, actually take place in the retina, might thereby be determined (see also section 65).

Helmholtz also applied his theory to explain other phenomena of colour vision, such as the accommodation of the visual organ to coloured light, symptoms of fatigue, after-images etc. In such explanations the supposition that each of the three fundamental processes can separately become "tired" and therefore decrease in sensitivity, plays a great part. While phenomena based purely on the laws of additive mixing in the eye can be completely explained by the Young-Helmholtz theory, this explanation has only a partial success with other phenomena. So many difficulties remain that it must really be said that the theory is quite inadequate to explain all the facts.

The experiments necessary for the quantitive support of the Young-Helmholtz theory are mainly the work of König. One of the first investigations (performed together with *Brodhun*) concerned the verification of the Grassmann laws [König [4])]. These appeared to hold accurately in the region of not too low a luminosity. König's [6]) chief work consists of the first accurate and very complete measurement of the spectrum which he performed in collaboration with *Dieterici*. He used spectral colours as primaries. As in this case negative quantities almost always occur and in both halves of the field of vision there must therefore be a mixture of two spectral colours. The Helmholtz-König "mixture apparatus" used for the experiments is especially constructed for this purpose. The most important part of this instrument is an equilateral prism which creates, as shown in fig. 65a, two adjacent halves of the field of vision, and at the same time fulfils the function of the prism P in fig. 69a, so that the pencils of rays 2 and 4 in fig. 65e each contain only two spectral colours which can be regulated at will, both in wavelength and quantity of light. For a description of the modern construction of this instrument see Richter [6]).

König did not restrict his measurements to normal observers; he also examined the various types of dichromats with the main purpose of establishing the isochromatic directions. He was the first to give a description of the rarely occurring defect of tritanopia, called by König [10]) "blue-blindness". He too attempted to trace racial differences in colour vision [König [3])]. Finally it must be mentioned

that K ö n i g [8]) was the creator of one of the first spectro-photometers in which the eye acts as a measuring instrument (section 50); this photometer is still frequently used in the improved form constructed by M a r t e n s [1]).

§ 73 *Other theories of colour vision*

The influence of the work of H e l m h o l t z and K ö n i g on the later developments of colour theory has been very great. The theoretical considerations gave rise to lively discussions and were the beginning of such an enormous number of new theories regarding colour vision that we cannot dream of sketching even the most important here. A few general remarks must suffice. For a short summary and list of references see S c h m i d t [2]).

Most theories apply particularly to one aspect of colour vision or to a particular group of experimental facts. For this group of phenomena they give a good explanation, but for phenomena remote from this group they appear to be unusable. The Y o u n g - H e l m h o l t z theory is no exception: the group of phenomena to which this theory is applied in the first place is that of additive colour mixture. Within this group such strict laws appear experimentally to exist that it is quite justifiable to take these into consideration from the start when working out a theory. But if we turn our attention to those phenomena of colour vision which have nothing to do with colour mixture it is frequently apparent that the theory is quite powerless; it even at first sight seems to be in conflict with the facts. This is particularly the case with phenomena of a more psychological nature. The following is a good example. If we approach the various colours of the spectrum with an open mind we come to the conclusion that there are *four* "main colours" each making the impression of forming a separate unit: red, yellow, green and blue. All other colours (orange, yellowish green, purple etc.) immediately suggest an association with the colours on either side, from which they can be produced by additive mixture. Everyone will recognize in orange a relationship with yellow and red, but hardly anyone will discover in yellow a combination of red and green. At first this fact of four psychological main colours appears to be contrary to the H e l m h o l t z - Y o u n g theory, which refers to three primary colours.

Such examples can be amplified and it is therefore not surprising that there was the most serious opposition to the opinions of H e l m-

holtz on the part of circles mainly concerned with the colour problem from the psychological side. The leading figure in those circles was *Hering*[1, 2]). There is a certain correspondence between the attitude of Goethe as a psychological protest against the opinions of Newton and the attitude of Hering towards Helmholtz Hering applied the above four main colours in his theory by accepting three processes: a red-green process, a yellow-blue process and a white-black process.

These processes are to be considered as chemical equilibria that can be displaced under the influence of light. Thus in the first process a certain equilibrium will make the sensation of "white", while displacement to the one side will produce the sensation of red and to the other side the sensation of green. The other two processes act in the same way. Various symptoms of colour vision — especially those for which the theory of Helmholtz was inadequate — can be better explained by the opinions of Hering, but.... the laws of additive colour mixture can only be explained with difficulty by this theory, while, for instance, the difference between protanopia and deuteranopia cannot really be understood according to Hering's ideas. In general the fact that there are three kinds of dichromats and three kinds of anomalous trichromats (see section 62) points strongly in the direction of Helmholtz's opinions.

The circumstances which we have just sketched were responsible for the fact that for a long time investigators were divided into two camps: adherents of theories closely related either to that of Helmholtz or that of Hering.

In retrospect this was truly an amazing phase in the history of colour science. Here were two conflicting theories, the principles of neither of which could directly be proved correct. Both camps were obliged to admit that a certain part of the phenomena could not be explained by them; the conflict consisted mainly in bringing forward certain phenomena which could be particularly well understood by means of the theory in question.

It was soon realized that the only possible solution was a fusion of the most important parts of both theories. This thought was most clearly uttered by v. Kries in his "Zône Theory" in which he accepted the fact that the processes in the retina act in accordance with Helmholtz's ideas, while Hering's theory obtains for a further link in the chain of processes which in the end produce the colour sensation. Von Kries expressed very fertile thoughts here, but his formulation is so vague that we should rather speak of a

"programme" than a "theory". Many later investigators have devoted themselves to the working out of this programme. Countless new theories have thus been built up and in a certain sense these new theories suffer just the same fate as the old ones: each theory has its group of phenomena for which it is especially suited!
The mass of solutions points to the fact that we are still far removed from a complete and final explanation of all phenomena. A first requirement for making progress and to make the theories less speculative is a better knowledge of the anatomical facts of the organ of sight.

Mathematically the bridge between Helmholtz's and Hering's ideas was spanned by Schrödinger[5]) and Luther[1]), who showed that transformations can be applied to the colour triangle such that the figures produced can be taken as an illustration of the four-colour theory of Hering [see also Richter[6])].

§ 74 *The further development of the experimental side of colorimetry*

Not only the theoretical considerations of Helmholtz but also the experimental researches of König have influenced the development of colorimetry for decades. From 1892 to 1931 almost all colour calculations were based either directly or indirectly on König's results. It was indeed realized in the course of time that even these excellent measurements were not perfect. Hence the fact that various later authors such as Ives[3]), Exner[1,2]), Weaver, [see Troland[1])], Judd[1]) etc. modified and "improved" the system designed by König, partly by combining it with their own or other measurements, partly by working them round in all manner of ways. Thus a number of more or less divergent systems arose of which we may mention the following: the original system of König-Dieterici, the König-Exner and König-Abney-Weaver system. This latter system arose from the fact that Weaver combined the values of König with those of Abney[1]), giving rise to the so-called O.S.A. system (Optical Society of America), which for six years has functioned as the official system in America and also found many users outside America. Gradually the undesirability of this state of affairs became more and more obvious. Not only was the existence of many systems that could not be accurately transposed into each other felt as a drawback, but the incompleteness of König's original measurements began to be felt. This incompleteness is mainly of two kinds: in the first place the measurements were carried out by only two

normal observers (König and Dieterici) so that it was doubtful whether the results could really be taken as that of the "average" normal observer; in the second place it became a known fact that König did not always work at a sufficiently high brightness so that rod vision was not completely excluded [this last drawback was pointed out by v. Kries[1, 5]) in 1897]. But it took till about 1930 before any considerable improvement was made in the state of affairs, when from two independent sources new determinations of the spectrum were published, both of which had been carried out with modern instruments and to the exclusion of the influence of rod vision. These were the measurements of Wright[2, 11]) (10 observers) and Guild[4]) (7 observers). Both investigators used quite different trichromatic colorimeters (of course of the subjective type). The fact that after transposition [see Guild[4])] the results of the two investigations showed a satisfactory correspondence (the O.S.A. figures differed at least six times as much from those of Guild, as did those of Guild and Wright from each other) formed an indication that the ideal was now not far off: the accurate determination of the mixing properties of the eye of the average normal observer.

Hence the average of the measurements of Wright and Guild were used, after recalculation to deduce a practically useful system. This is the system recommended by the C.I.E. in 1931, a system which in the course of the following decade replaced the old systems everywhere and on which all the calculations and tables in this book have been based.

That the measurements of König were very deeply rooted appears from the fact that Schouten[1]), for instance, used these measurements as late as 1935 in thinking out a new quantitive (!) "four-colour theory", just as Ströbl[1]) did in his dissertation in 1937 and in a publication of 1939. Such examples are known in still more recent times [see Willmer[1]), who uses Abney's 1895 curve], but it can be said that by about 1940 the C.I.E. system had been universally adopted.

Together with the adoption of the new determination of the spectrum came the introduction of the "white" standard illuminants A, B and C (section 23) whereby a certain unity has also been obtained internationally in the measurement of coloured surfaces.

An increasing understanding has also been attained in the last 30 years of the foundations on which colorimetry is based. In this respect the excellent work of Schrödinger[2, 7]) in particular must be mentioned, who in contrast to most of his contemporaries again brought *colour space* to the fore and thus arrived at a clear and

logical method of explaining the basic facts. On broad lines I have followed the example of Schrödinger in the preceding chapters.

§ 75 *Wilhelm Ostwald (1853-1932)*

It can be said that the development of the science of colorimetry has moved in a fairly straight line from Newton via Young, Maxwell, Helmholtz, König to the modern investigators such as Schrödinger, Wright, Guild etc.
However, just as Goethe, at the beginning of the previous century, deliberately abandoned this straight path, so have others in this century, for instance in the direction of the *Ostwald* colour theory.
There is in some respects a correspondence between these two wanderers from the straight path. Just as Goethe turned against Newton and considered, quite incorrectly, his ideas as quite wrong and valueless, so Ostwald (and particularly his disciples!) turned against the school of Helmholtz. Here he fell into the error already mentioned in section 72 of renouncing the whole work of Helmholtz and König, losing sight of the fact that a great part of this work is firmly based on undeniable experimental facts. Another remarkable coincidence exists in the fact that Ostwald, like Goethe, started from boundary colours or combinations of boundary colours.
Ostwald's[1, 2, 3] "colour theory" is, to use von Kries' words, "eine glückliche Systematisierung der Körperfarben" (an excellent classification of surface colours), therefore a method of obtaining a general view of all colours that surfaces may assume. In some respects this method appears to be more demonstrable than the method dealt with so far (the assignation of *XYZ* values). We know that the colours of the objects depend on the nature of the illuminant used: Ostwald works only with sunlight in his speculations.
The starting point of his colour theory is the so-called "Vollfarben" (full colours) or C-colours whose properties we have discussed extensively in section 48. In fig. 59 the position of these colours in the colour triangle is given (with a standard white B as illuminant). Ostwald considered these colours as the most "colourful" among all possible colour sensations and supposed that all surface colours occurring in practice have their colour points within the curve drawn in fig. 59.

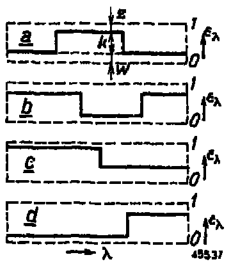


Fig. 90
Schematization of surface colours according to Ostwald.

Now the C-colours form a special case of the ideal colours, and they have therefore a spectral reflection curve of one of the types sketched in fig. 51, in which, in cases *a* and *b*, the two limiting wavelengths are complementary. Further, Ostwald took as an accepted fact that all colours occurring in practice are either C-colours or can be deduced from C-colours, by replacing the reflection factors 0 and 1 by other constants. In other words, that all surface colours show reflection curves approximately like one of the types sketched in *fig. 90*. From this assumption it follows directly that each surface colour is related to a certain C-colour (produced by replacing the constant reflection factor by 1 and 0) and that we can imagine any surface colour to be produced by mixing that C-colour and the standard white in a certain ratio. The C-colour therefore here plays exactly the same part as the dominant wavelength in the monochromatic system: the surface colour always lies on the connecting line between white point and appropriate C-colour in the colour triangle. Now we can completely establish a surface colour with a reflection curve as given in fig. 90, by quoting the two (or one) wavelengths at which the reflection curve makes a sudden jump (by this the appropriate C-colour has already been specified) and further by giving the two constant reflection factors. In Ostwald's system, however, the C-colour is not as a rule fixed by the two complementary wavelengths but by a certain number N; the C-colours lying on the curve of fig. 59 must therefore be provided with an identification number. Ostwald let these numbers run from $N = 0$ to $N = 100$, later from 0 to 24. The value of N therefore plays an analogous part here to λ_d in the monochromatic system. The positions of the numbers on the curve were fixed by the "Prinzip der inneren Symmetrie" ("Principle of inner symmetry"). This is as follows: when additive mixing of the C-colours N and $(N + 2)$ takes place a colour is produced with the same dominant wavelength as the C-colour $(N + 1)$. The colour $N + 12$ then automatically becomes complementary to N. This principle enables us to calculate exactly the distribution of the C-colours over the curve. In *fig. 91* the result of this calculation has been given.

The calculation is the work of Bouma[14]). An earlier calculation by Richter[10]) makes rather arbitrary use of other instructions given by Ostwald which partially contradict each other.

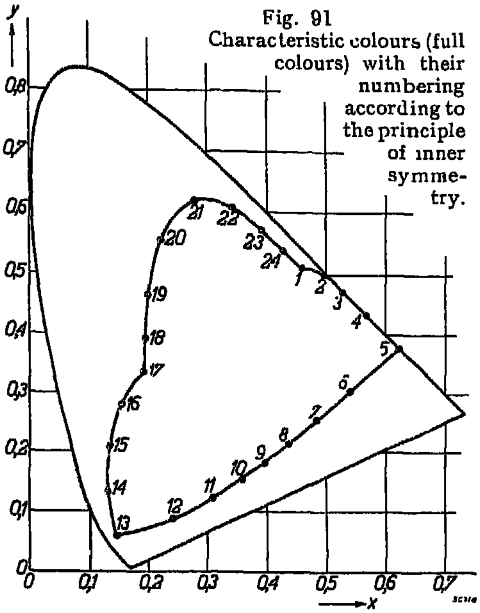


Fig. 91
Characteristic colours (full colours) with their numbering according to the principle of inner symmetry.

In order to distinguish all colours with the same dominant wavelength (therefore having the same N), Ostwald fixed the two constant reflection factors as shown in fig. 90 by the three magnitudes k, w and z. These three letters are used to indicate the colour content, white content and black content. The logic of this nomenclature can best be seen by imagining the colour to be reproduced with the aid of a revolving sector disc (see also fig. 70). Place the colour to be examined in the central circle (which colour is supposed to be of the type of fig. 90); the match can then be obtained by dividing the outer ring into the following sectors *(fig. 92)*:

$k \cdot 360°$ of the appropriafe C-colour
$w \cdot 360°$ of the brightest white : 1
$z \cdot 360°$ of the deepest black : 0

We can see from the reflection curve of fig. 90 that we can indeed imagine the colour to be compounded of the given parts of white and the appropriate C-colour. For the wavelengths not reflected by the C-colours the light proceeds partly from w and partly from k, so that part $w + k$ is reflected. We therefore get exactly the reflection factors indicated in fig. 90a for all wavelengths.

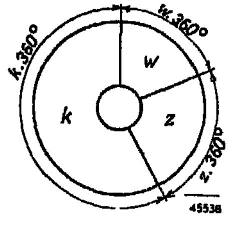


Fig. 92
The meaning of the terms colour content (k) white content (w) and black content (z) in connection with the sectored disc.

Of course fig. 92 only represents an imaginary experiment, for the colours required by the outer circle are not to be realized without special aids and appliances. If we should, however, succeed in carrying out the experiment correctly, the match would be correct *for* all trichromats, dichromats and monochromats, for we have not only reproduced the colour sensation but also the spectral reflection curve!

According to O s t w a l d's system a colour is therefore characterized by the quantities N, k, w and z (in which clearly $k + w + z = 1$). On this system O s t w a l d's [1, 2]) well-known colour atlas is based. The values of these four quantities are marked on each of the coloured cards (more than 2,000 in his first edition).

In the abridged form of O s t w a l d's atlas every "page" (N = const., therefore constant λ_d) contains the grey scale (8 steps) and 28 coloured chips.

These 36 colours are denoted by the following symbols:

							a
						ca	
					ea		c
				ga		ec	
			ia		gc		e
		la		ic		ge	
	na		lc		ie		g
pa		nc		le		ig	
	pc		ne		lg		i
		pe		ng		li	
			pg		ni		l
				pi		nl	
					pl		n
						pn	
							p

In each combination of letters, such as ne, the first (n) denotes the white part, in percent:

	a	c	e	g	i	l	n	p
w	89	56	35	22	14	8.9	5.6	3.5

the eight numbers forming a geometrical progression in accordance with "W e b e r's law" (section 78).

The second letter (c) denotes the black part z in per cent:

	a	c	e	g	i	l	n	p
z	11	44	65	78	86	91.1	95.4	96.5

The corresponding numbers in the two tables add up to 100, *e.g.* for (a): $89 + 11 = 100$, etc.

Both tables also apply to the grey scale. So a grey colour denoted by (i) has a white content of 14 per cent and a black content of 86 per cent. For the colour ne: $w = 5.6$, $z = 65$ and therefore $k = 29.4$ ($5.6 + 65 + 29.4 = 100$).

For the red and yellow hues the notation corresponds reasonably well to reality; for the bluish hues, however, the black part is in reality higher (up to 10 per cent) than corresponds to the inscription on the chip.
This adjustment was made in favour of Ostwald's colour harmony laws, according to which simple relations should exist between the letter combinations of harmonising colours, a subject which occupied an important place in Ostwald's considerations on colours, upon which however, we shall not enter here (section 77 E).

In addition Ostwald has given a simple method of determining these quantities: N is measured by comparison with a series of 100 (or 24) cards of which N is fixed once and for all with the aid of the "Prinzip der inneren Symmetrie". ("Principle of inner symmetry"). For measuring k, w and z for colours of the type of fig. 90 it is sufficient to measure the reflection factor in two narrow spectral bands: if the bands are well selected, the one factor gives w and the other $k + w = 1 - z$. The measurement can be made with the aid of filters which only transmit a narrow part of the spectrum.

In the above sketch of Ostwald's colour system many intrinsic contradictions and vague ideas have been omitted and the rather inexact form of the original speculations has been replaced by the language now usual in works on colorimetry.

The *advantages* of Ostwald's system, to which it owes its popularity in certain circles, are obvious:

a. The definition in terms of N, k, w and z is much more acceptable and easier for the layman to understand than that in terms of X, Y and Z: not much imaginative power is required to "see" immediately the large proportion of black in a dark brown colour and the high proportion of white in a light pink colour.

b. The methods of measurement are very simple, whether we specify the quantities by direct comparison with the colour atlas, or whether we determine k, w and z by means of filters in the manner described above.

There are, however, very serious *disadvantages* attached to Ostwald's system which partly originate from the fact that the assumptions made by Ostwald are not all correct. For instance:

a. The assumption that all colours belong to one of the types of fig. 90 is incorrect. Only in a limited number of cases is there a reflection curve of approximately those types. Now the theory can indeed be saved by adding the stipulation that two colours with the same appearance must have the same N, k, w and z (whereby these quantities are also fixed for any other given colour), but it is at once clear that we can no longer apply the

simple method given by Ostwald for the specification of black content and white content. By this one of the greatest advantages of the system is lost.

b. The assumption that all colours lie within the curve of the C-colours is incorrect. Many colours occur in practice, especially among the reds, which lie outside that curve. The consequence of this fact is reflected in the colour atlas: certain shades of very saturated pure red are completely missing. In an attempt to save the theory (which here leads to a negative white content) the advantage of demonstrability is lost here also.

c. As a result of the incorrect methods of measurement used in making the cards of the colour atlas, the quantities given on the cards do not coincide with those calculated by the theory as above extended [see (a)] to arbitrary colours.

d. All the drawbacks already discussed in section 56 apply to the use of the colour atlas.

Considering the advantages and disadvantages just brought forward it need not be found surprising that the Ostwald colour theory has been the subject of lengthy discussions.
Opponents, von Kries[11]), K. W. F. Kohlrausch[1]) Schrödinger[6]), Schäfer[1]), Bernays[1]) among others and supporters, Meissner[1]), Seitz[1]), Oryng[1]), Schollmeyer[1]) among others pursue one another, while in more recent years an attempt was made to extend and correct the theory in such a way that the worst contradictions disappeared, so that a connection could be made between Ostwald coordinates and trichromatic coördinates [Miescher[1]), Luther[1]), Richter[1, 9, 10])].
In this way it is hoped to save Ostwald's colour theory for those subjects in which the advantages exceed the disadvantages.

See also Zeishold[1]), Foss[1]), Granville[1]), Birren[1]).

§ 75a *Albert H. Munsell (1858-1918)*

* *A. H. Munsell* was an artist and an art teacher. To aid his teaching he devised a system of colour-order which is in principle a monochromatic system in which the colours are distinguished by their lightness, hue and saturation (called "value" *(V)*, "hue" *(H)* and "chroma" *(C)* by Munsell).
Although originally an artist, Munsell was enough of a physicist to understand the principles of photometry and colorimetry. As a

result his system was partially founded on physical principles. For instance the reflectivity ε_r relative to MgO ($\varepsilon_r = \varepsilon/\varepsilon_{MgO}$) was correlated with the "value". A scale of ten apparently equal value steps was painted (painting went first!) and then by photometric measurement it was found out that approximately

$$V^2 = \varepsilon_r \cdot 100$$

so $\varepsilon_r = 0.36$ would correspond to $V = 6$.
The correspondence which is now generally accepted is empirical:

V	1	2	3	4	5	6	7	8	9	(10)
$100\,\varepsilon_r$	1.21	3.13	6.56	12.0	19.77	30.05	43.06	59.10	78.66	(102.56)

For the fixing of the hue-number, the colour circle (the range of spectral colours, completed by the purple line) was divided into ten "equal" steps by first choosing five principle hues R (red), Y (yellow), G (green), B (blue) and P (purple), and then 5 intermediate hues RY, YG etc. These hues were again found empirically by comparing painted chips, but they were to comply with the requirement that two hues 5 numbers apart in the scale should be complementary (e.g. R and GB). Each hue step may be divided into 10 smaller steps (R ranging from $1R$ to $10R$, for instance).
Finally every hue was painted in a number of apparently equal "chroma" steps for each value. Colours of equal chroma but different hue or value should show equal apparent saturation.
A colour is indicated by $H\ V/C$, *e.g.* $5R\ 8/6$, where $5R$ denotes the hue, 8 the value number and 6 the chroma number.

Originally Munsell applied a physical principle to chroma in consequence of which equal areas of two complementary colours having the same VC product should add to grey in a disc mixture test. In consequence of this, for any hue, the relation between colorimetric purity, value and chroma should be

$$p_{V/C} = (C/V)\ p_{5/5}$$

whereas colours having the same value of the ratio V/C should have the same chromaticity (x, y) [Gibson[9]) and Nickerson].

We thus see Munsell following up two trains of thought at the same time, one psycho-physical, based on physical concepts (ε, λ_d, p), the other psychological, based on empirically determined apparently equal steps.
The psychological problem of colour differences has not been discussed in the foregoing chapters. This will be done in chapter XIII and the reader is referred to this chapter for further information. As the simultaneous pursuit of the two lines of thought at the same time proved impossible, some of the psycho-physical requirements

(especially those referring to chroma) have been dropped, the system gradually growing into an almost purely psychological system of colour order. By the cooperation of the Munsell Colour Company[1]), the Colour Council and the Nat. Bur. of Standards, the tristimulus values of the Munsell Book of Colour have been measured and calculated for a number of illuminations, so that the colours may be located in the C.I.E. chromaticity diagram [Kelly [1])].

From the charts given in this publication it may be seen that the requirement of complementarity is only approximately fulfilled. Neither are the lines of constant hue straight lines passing through the white point. The lines of constant chroma, although fairly regular, show some irregularities.

At the same time the psychological requirement of equality of consecutive steps has been tested by a number of persons [Newhall [3])]. This has resulted in a preliminary renotation of the existing samples, using decimal numbers. By using these numbers smooth curves of constant hue and chroma can be drawn in the chromaticity diagram for each value [see also Granville [2]) and Nickerson [1,4])]. The lines of constant hue are not straight and not the same for different values. The lines of constant chroma in no way coincide with the lines p = const or σ = const (sect. 83).

Ostwald met Munsell in 1905 and exchanged ideas with him. In 1913 and 1914 Munsell went to Europe to give an exposition of his ideas. Wide-spread use is made of the Munsell colour atlas in America [revised by Munsell's son A. E. O. Munsell[1])] and there is an increasing interest for this system in other parts of the world [see Judd [14])].

CHAPTER XII

Discrimination

§ 76 *Other tasks for the eye*

In the previous chapters we have constructed a kind of "colour geometry" and have learnt to classify and measure colours. All these speculations are based on experiments in which one of two simultaneously displayed spots of light was varied until both spots were indistinguishable. Here the sole task of the eye was to judge whether both colour sensations were equal.

Besides the many colour problems which can be solved by experiments of this kind there are, however, a number of others in which the human organ of sight does indeed act as judge but has another and usually more difficult task to fulfil than that of a guide in adjusting to equality. Some of these tasks are as follows:

A. Discrimination. When a good adjustment to equality has been made it appears to be possible slightly to alter one of the colours to be compared without the eye being able to observe the difference. It is often important to know how far such an alteration can go before the eye begins to notice a difference in colour. In the measurements that should give an answer to this question it is usual to start with two halves of the field of vision which not only look equal but are completely identical in a physical sense as well, meaning of course as regards radiated power and spectral composition. Then the amount by which one of the two has to be altered to make a noticeable difference, the so called "discrimination step", "threshold" or "limen" is measured. The alteration is usually such that a certain property of the colour is varied while leaving the other properties as far as possible unaltered. For example, the brightness alone is varied, the spectral composition and the chromaticity left unaltered; or we begin with a spectral colour and change the wavelength (leaving the brightness constant). In the first case we speak of the brightness-difference limen, in the second case of the wavelength-difference limen. More generally we can say that the change always corresponds to a movement through the colour space along a particular curve. In the first example this curve was a

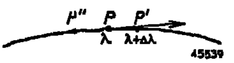


Fig. 93
Limen for a given direction in colour space. Along the curve the colour can move from P to P′ without a difference being observed.

straight line through the origin, in the second example the curve of the spectral colours in a plane of constant brightness.

As the alterations are as a rule very small we can replace the curve by its tangent at the starting point. We may then speak of the limen or just noticeable step for a certain *direction* in colour space: see *fig. 93* which illustrates the second example: the curve represents the curve of the spectral colours in a plane of constant brightness B. P is the starting point of the experiment (spectral colour with wavelength λ and brightness B). P′ is the point to which we can move before a colour change is observed (spectral colour with wavelength $\lambda + \Delta\lambda$ and brightness B); $\Delta\lambda$ is the limen for the wavelength difference, while the arrow gives the direction in the plane of constant brightness in which the limen is determined by the experiment.

Among other things a knowledge of discrimination steps is important for judging the accuracy to be attained in visual colorimetry and photometry; for the requirements with which different coloured objects must comply in order to appear to match the colour; for the minimum differences in brightness required for an object to show against a background (street lighting) etc. The task to be performed by the eye in determining the limen agrees fairly closely with the task with which the eye was formerly charged. The difference is that formerly we moved along a curve (see fig. 93) in colour space and tried to determine the centre of the range P″P′ in which no difference in colour sensation appeared, while in determining the limen we are concerned with fixing the distance from P″ to P′, or the size of the interval P″P′.

B. The selection of a sample in a set of colours most resembling a given colour in a certain respect.

As a first example of an experiment in which the eye has to perform such a task we can take the visual adjustment to equal apparent brightness if we have two different colours to compare (heterochromatic photometry, see section 11). Here one of the colours is left unaltered while we make the other pass through a series in which the brightness is variable, while the chromaticity is kept constant. We must now choose the sample from this series which agrees in apparent brightness with the colour in the other half of the field of vision.

As a second example we will take the adjustment to equal saturations of two lights differing in dominant wavelength and therefore in hue. This example is analogous to the first. We leave one colour unaltered while we make the other pass through a series in which the saturation is varied but the brightness and dominant wavelength are kept constant; we must choose from this series the sample agreeing in saturation with the colour kept constant. The essential difference from the former tasks is that here among the series of colour sensations from which we must make a choice there is no sample that is indistinguishable from the constant colour sensation in the other half of the field of vision. The result will therefore always be that two colours are chosen showing mutual differences but which agree in a particular property (for instance apparent brightness, saturation or hue).

The changes to which we subject one of the colours during adjustment again resolve themselves into this, that we make a colour pass along a certain curve in colour space (in heterochromatic photometry a straight line through the origin). In contradistinction to case A the colour point of the unaltered light impression of the other half of the field of vision does not lie on the curve.

C. Determining the extent to which two colour sensations differ.

First example: Let two light sensations be given with the same relative spectral composition but considerable differences in brightness. Required: the adjustment of a third colour so that the chromaticity corresponds with the two other colours, while the apparent brightness of this third colour differs just as widely from that of the first colour as from that of the second.

Second example: Given a "white" colour (for instance, standard illuminant A) and a green colour with a colorimetric purity of say $p = 1/2$, both colours have a brightness B. Required: to adjust the colorimetric purity p' of a third colour whose dominant wavelength and brightness must amount to 589 mμ and B respectively, so that this third colour differs as much from the white colour as from the green.

We therefore see that with this type of adjustment at least three colours are involved, while only one of these is varied; the variation again takes place along a curve in colour space.

In general it may be said that in passing from the simple equality adjustment to processes A, B and C the adjustment becomes continuously more difficult and uncertain, while the result is

increasingly influenced by attendant circumstances and the individual properties of the observer.

D. The naming of colours. The simplest case is that in which only one colour (for instance a spectral colour) is shown in the field of vision and the observer is asked to judge whether he would call the colour green, yellowish green, yellow green, greenish yellow, yellow etc. Often, beside the colour to be named, another colour is present in the field of vision, for instance in the form of coloured surroundings. Tests of this kind are important in cases where signal colours have to be recognized and in studying the influence of environment on the colour sensation and the condition of the retina.

E. Specifying colour combinations pleasant to the eye. This problem lies outside the scope of this book. It is undoubtedly rather complicated and can only be solved by an artist and a physicist in collaboration. The artist must create or indicate the desired combinations and afterwards they must be measured by the physicist, who will try to deduce general rules from the results. This field of research is still for the greater part unexplored.

§ 77 *Brightness discrimination*

As has already been remarked, the sensitivity to brightness differences is important for several very divergent purposes, and it need not therefore be surprising that many investigators have carried out measurements in this respect.

It may seem strange at first sight that the results of these measurements are very divergent and that a great many misconceptions exist in this field.

Some of these results are given in *fig. 94.* As the range of brightness over which the measurements extend is so enormously large, a logarithmic scale has been used for brightness B. (If we pass along

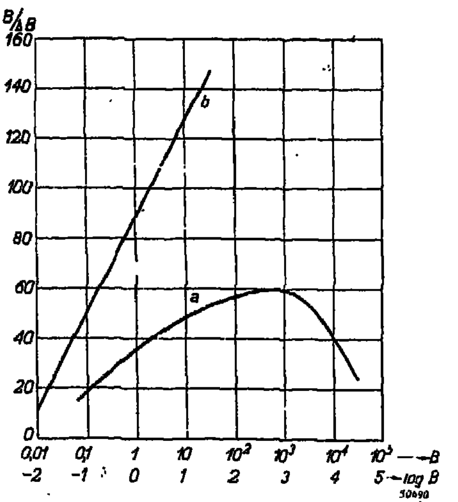


Fig. 94
Sensitivity $B/\Delta B$ for differences in brightness: a brightness $B + \Delta B$ can just be distinguished from a brightness B. a) Dark surroundings, b) light surroundings.

the horizontal axis, log B increases regularly while B increases every time by a factor 10 when proceeding over equal distances.) To the brightness B we must now add a brightness ΔB (the limen), necessary to see a distinction. This means a relative increase of $\frac{\Delta B}{B}$. This quantity is therefore a measure of the sensitivity of the eye for small differences of brightness. As, however, a low value of $\Delta B/B$ points to a high sensitivity (a small relative luminosity increase can be observed) we prefer to plot the inverse value $\frac{B}{\Delta B}$, as has been done in fig. 94, and call $B/\Delta B$ the sensitivity of the eye to differences of brightness.

The name "contrast sensitivity", often used for this concept, is less happily chosen for it may cause confusion in connection with the fact that the physiologist understands something quite different by "contrast" than do the illuminating engineer and the photographer.

Now in fig. 94 curve *a* represents the average results of the measurements of König[5]), Blanchard[1]) and the author, which agree well with each other; curve *b* gives Stile's'[1]) measurements. The great difference in the two curves is due to the conditions under which the measurements were carried out. In the measurements resulting in curve *a* the two light spots to be compared only occupied a relatively small part of the field of vision, while the remainder was dark. In the tests from which curve *b* resulted the whole field of vision showed a uniform brightness B, while subsequently in a part of the field of vision the brightness was increased to $B + \Delta B$.

It is clear from fig. 94 what an enormous influence certain accessory conditions can have on the results of discrimination measurements (for instance the illumination of the surroundings). In general it may be said that the processes described in section 76 under A, B, C and D are much more sensitive to such influences than is the case with simple equality adjustments.

The curves *a* and *b* of fig. 94 both have their practical interest. In some cases conditions occur in practice corresponding to those of curve *a* (observations on artificially lighted roads, the use of searchlights, various photometric and colorimetric instruments etc.). In other cases conditions resemble more those giving rise to curve *b* (observations in the open country, especially with snow-covered landscapes, diffusely lighted rooms with light walls etc.).

In many cases conditions lie midway between these extremes: the environment is well lit but far less brightly than the middle of the

field of vision where observations are carried out. In such cases values for the sensitivity of the eye lying in between *a* and *b* must be expected.

Finally it may happen that the environment or a part of the environment displays a considerably greater brightness than the field in which we wish to observe small differences of brightness. In that case the sensitivity of the eye to differences of brightness is greatly decreased by the disturbing presence of the high brightness. We then speak of *glare*: this makes all discrimination steps greater. Examples are: the presence of an illuminant of too great brightness in the field of vision; the distinguishing of very dark colours in a light environment, etc.

The shapes of curves *a* and *b* are also influenced by other factors. For instance, the size of the surfaces to be compared and the colour of the light. The first influence makes itself felt particularly for objects seen at a very acute angle (less than one degree for instance). The influence of colour is mainly felt at low brightnesses when the Purkinje effect occurs. [For sensitivity to differences of brightness under radically different conditions, see for instance A. E. O. Munsell[2]), Sloan and Godlove].

It is obvious from the above considerations that we cannot speak of a general connection between $B/\Delta B$ and B. Yet there is a widespread opinion that such a simple and generally obtaining connection should exist. This opinion is formulated in the form of Weber's law ("Psychophysisches Fundamentalgesetz") according to which the ratio $\frac{B}{\Delta B}$ should be constant, or in other words: the limen ΔB should be a fixed percentage of the original brightness B.

The much-used name "Weber-Fechner law" is less exact. The law originated with Weber (1834), whilst Fechner (1858) drew the very questionable conclusion from it that the brightness sensation must be proportional to the logarithm of the brightness B [see the extensive, witty but not always well founded criticisms of Moon[1]) and further Cobb[1]) and Guild[5])].

It appears from fig. 94 to what extent the Weber law holds. Under the conditions of curve *a* (dark surroundings) the sensitivity to differences of brightness in the luminosity range of 10 to 5,000 c/m² *) is very approximately constant (53 ± 5). In this luminosity range (comprising most daytime brightnesses and those of illuminated living rooms) Weber's law is useful under the

*) 1 c/m² = 1 candle per square meter = $\frac{1}{10{,}000}$ c/cm² = $\frac{1}{10{,}000}$ stilb (sect. 17).

stated conditions as an approximation. In the luminosity range with which we are concerned in street lighting the law does not hold at all.

§ 78 *Hue discrimination*

Hue discrimination, although of less practical importance than brightness discrimination, has also been studied by innumerable investigators.

Fig. 95 gives some results. If a spectral colour λ can just be

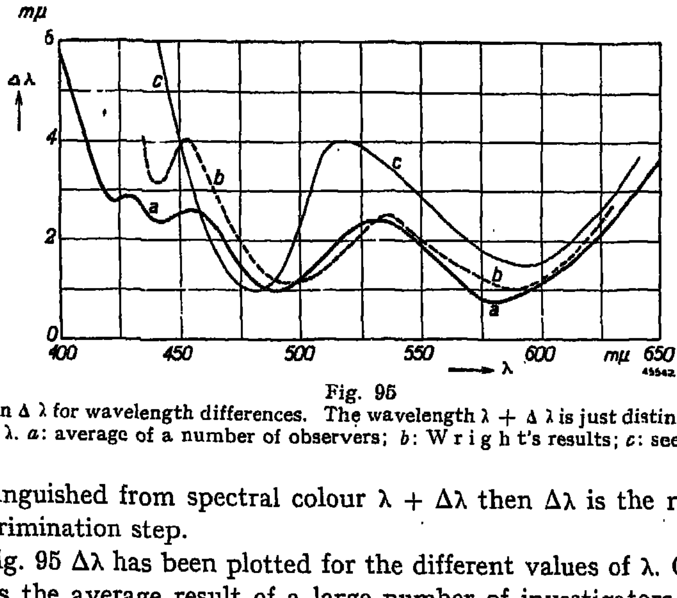


Fig. 95
Limen Δ λ for wavelength differences. The wavelength λ + Δ λ is just distinguishable from λ. *a*: average of a number of observers; *b*: Wright's results; *c*: see sect. 81.

distinguished from spectral colour λ + Δλ then Δλ is the required discrimination step.

In fig. 95 Δλ has been plotted for the different values of λ. Curve *a* gives the average result of a large number of investigators [Uhthoff[1]), König[2]), Dieterici, Exner[1]), Steindler[1]), Jones[1]), Laurens[1]), Rosencrantz[1]), Sachs[1]), Tyndall[1]), Wright[3]), Möller Ladekarl[1]), Haase[1])] while curve *b* gives the results of investigations which Wright[5]) carried out on his own eye.

In these investigations, too, all kinds of accessory circumstances appear to have exercised a considerable influence on the result. The results of the individual investigators are consequently rather widely divergent, especially in the blue and violet, while the various observers who carried out measurements with the aid of the same apparatus, and therefore under identical conditions, show a much better agreement among each other.

The results are further influenced by the brightness and the size of

the coloured light spots to be compared, the environment (which for most experiments was dark), etc. Moreover another source of divergence can occur here. As it is particularly desired to measure the sensitivity of the eye to wavelength differences of the spectral colours the brightnesses of the two halves of the field of vision must always be kept equal: a slight difference in brightness can have a strong influence on the results. To maintain this equality of brightness may be rather difficult, and several investigators have not paid sufficient attention to this point, so that they sometimes actually measured a combination of the sensitivity to wavelength differences and to differences in brightness. In other words, they measured the limen in the wrong direction in colour space. The part of curve *a* (fig. 95) for the wavelengths smaller than 450 mμ is particularly unreliable in consequence of this source of errors.

The sensitivity to wavelength differences has also been studied repeatedly for people with defective colour vision (see chapter X), and it has been found that most of such persons manifest a lower sensitivity, therefore a larger discrimination step, than normal observers. We shall not discuss this further. Extensive investigations on this point are to be found in Pitt[1]) (dichromats), Nelson[1]) (anomalous trichromats) and innumerable other authors [see Brodhun[1]), Steindler[1]), Rosencrantz[1]), Sachs[1]), Möller Ladekarl[1]), Corbett[1]), Engelking[2])].

With the help of a curve giving $\Delta\lambda$ as a function of λ we can estimate the number of colours distinguishable in the spectrum by the observer. For instance, from curve *a* of fig. 95 about 152 are found. (Of course not all the colours can be named, but when placed side by side the difference can be seen.)

If we do not start from the spectral colours but from a series of colours of equal brightness B and colorimetric purity p but varying dominant wavelength λ_d, the sensitivity of the eye can be determined for the differences $\Delta\lambda$. We therefore measure the limens along curves of constant p (see fig. 31). Such measurements have been carried out, for instance, by Tyndall[1]) and Haase[1]). Both investigators come to the rather unexpected result that p can be made to fall from 1.0 to about 0.5 before the curve giving $\Delta\lambda$ as a function of λ alters perceptibly. Hence it follows that almost as many colours can be distinguished along the curve $p = 0.50$ (fig. 31) as along the curve of the spectral colours ($p = 1$). If the colorimetric purity is allowed to fall still further this number of distinguishable colours rapidly decreases. This is of course self-evident. If we describe

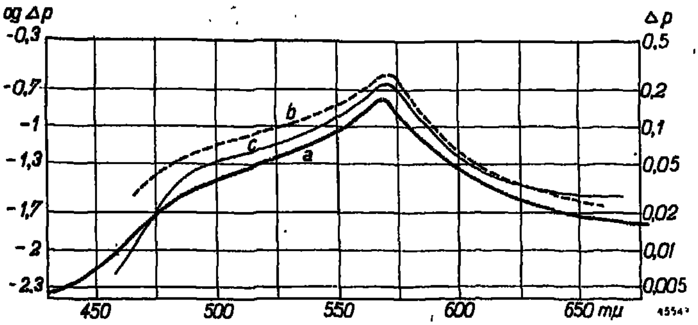


Fig. 96

Thresholds Δp for differences in purity. The colour having a purity Δp can just be distinguished from white ($p = 0$). *a:* Average of a number of observers. *b:* Wright's results. *c:* See sect. 81.

a curve of constant but very low p we always remain in the immediate neighbourhood of the white point, so that hardly any difference in colour is observed.

§ 79 *Other limens and thresholds* *)

If the limens for colorimetric purity are required we must move in a plane of constant brightness along a straight line running through the white point (one of the straight lines drawn in fig. 31 through E) and measure the extent of such a movement before a difference in apparent saturation is observed. Many workers have occupied themselves exclusively with the beginning and the end of this range. In other words they have either measured how far one must be from the white point before a faint difference is visible or how far one must be from the spectral locus (in the direction of the white point) in order to be able just to notice a difference from the spectral colours.
In the first case, therefore, a white light spot (brightness B) is compared with a mixture of a certain spectral colour (brightness B_S) with white light (brightness $B - B_S$) and the threshold of B_S determined. The colorimetric purity of one half of the field of vision is therefore $p = 0$ and that of the other half $p = B_S : B$. We can therefore just distinguish a difference in colorimetric purity of an amount B / B_S.

*) Throughout this book the smallest perceptible value of any quantity is called *threshold*, whereas the perceptible increment of any quantity is called difference threshold or *limen*. This practice is purely conventional, for the meaning of the latin word "limen" (plural in English: limens) is also "threshold". Some authors prefer circumscriptions as "just noticeable difference" etc.

If the sensitivity of the eye to differences in colorimetric purity is to be studied, the inverse value of this threshold, $B : B_S$, comes into consideration.

Fig. 96 shows the results of some investigators who have measured the threshold $\Delta p = B_S : B$ along different lines through the white point (therefore for different wavelengths λ of the spectral colour used in the mixture). As the values of Δp vary very considerably with wavelength (up to a factor 40!) we have plotted the threshold Δp logarithmically.

On the left is the scale of log (Δp) (regular scale division), on the right Δp itself (irregular division).

Curve *a* gives the average result of a number of investigators [Jones[3]), Purdy[1]), Warburton[1]), Martin[1]), Wright[6]), Nelson[2]), Priest[5])] while curve *b* gives the result of the investigations which Wright[6]) carried out on his own eye. It will be seen that the curve displays a pronounced maximum for a wavelength of 570 mμ. That is to say, for this yellow colour the threshold is highest and the eye is therefore least sensitive to differences in colorimetric purity. For this wavelength about 1/6 of the white light must be replaced by the spectral colour before a difference in colour is noticeable, while for red (680 mμ) the admixture of 1/60 part is already sufficient!

The results of the various authors quoted are rather widely divergent, as a result of different accessory circumstances (chiefly the size of the field of vision), but these causes increase the thresholds for all wavelengths by about the same factor. If we therefore plot the results on a logarithmic scale (fig. 96) the various curves are shifted relative to each other but have the same shape. This is shown very well by curves *a* and *b*.

In the second case mentioned at the beginning of this section a spectral colour (brightness B) is compared with a mixture consisting of a small part of white (brightness B_W) and the remainder spectral colour; the threshold of B_W is determined. The colorimetric purity of one half of the field of vision is then 1, and in the other half $p = 1 - B_W/B$ the difference is therefore $B_W : B$; in this case a difference in colorimetric purity of $\Delta p = B_W/B$ can just be distinguished. *Fig. 97* shows some experimental results: *a* the average of several investigators [Jones[3]), Wright[5]), Warburton[1])]; *b* the values of Wright[5]).

In contrast to fig. 96 we see that here the results depend only slightly on the wavelength. The eye is about equally sensitive to saturation differences for all points in the neighbourhood of the curve of the spectral colours. For intermediate cases — sensitivity to saturation

differences starting from colour points lying between the white point and the curve of the spectral colours — fewer reliable measurements have been carried out. The following general conclusions can be drawn from the measurements of Jones[3]) and Warburton[1]): if a straight line from the white point to the spectral locus is traversed

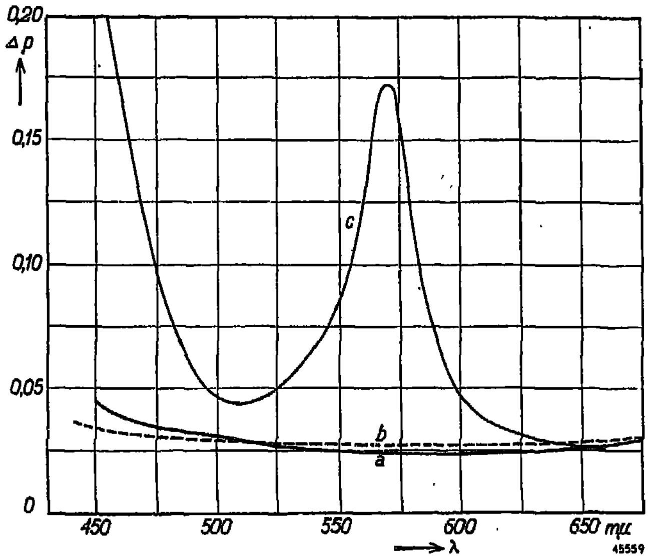


Fig. 97

Thresholds Δp for differences in purity: the colour having a purity $1 - \Delta p$ can just be distinguished from a spectral colour ($p = 1$). *a:* Average of a number of observers. *b:* Wright's results. *c:* See sect 81.

(if, therefore, the colorimetric purity p is increased from 0 to 1 with a constant dominant wavelength) the sensitivity to saturation differences decreases at first, reaches a flat minimum between $p = 0.4$ and $p = 0.6$ (here the sensitivity has decreased by a factor 2 — 3) and then rises again to the final value for $p = 1$ (see fig. 97). The limen for saturation differences is therefore greatest for colours of medium saturation. Other investigators [Donath[1])], however, found a somewhat divergent sensitivity curve.

Just as the total number of spectral colours was calculated from the limens for wavelength differences (sect. 78), the total number of steps of just noticeable colour differences along a straight line from the white point to the curve of the spectral colours can be counted if the magnitude of the limen for saturation differences is known for each point of the line.

In *fig. 98* curve *a* shows some results for different values of the dominant wavelength λ_d [average of the careful measurements of W a r b u r t o n[1]) and M a r t i n[1]) and the less extensive measurements of J o n e s[3]) and L o w r y]. This curve shows a minimum where that of fig. 96 had a maximum (570 mμ). This was to be expected; for this wavelength the sensitivity to differences of saturation was particularly low, so that we can only distinguish a small number of colours between the white point and this yellow spectral colour. Besides the limens for differences in brightness, wavelength and colorimetric purity, measurements of just noticeable steps along other curves have been carried out, for instance, along the curve of temperature radiators [J u d d[6]), P r i e s t[3])] and through a series of coloured filter glasses [J u d d[5, 7])], but we shall pay no further attention to these investigations.

§ 80 *Attempts to summarize all discrimination measurements*

In sect. 77-78 a number of investigations in the field of discrimination and the sensitivity of the eye to colour differences have been described. Of course the number of these investigations can be considerably extended and limens could be measured along the most

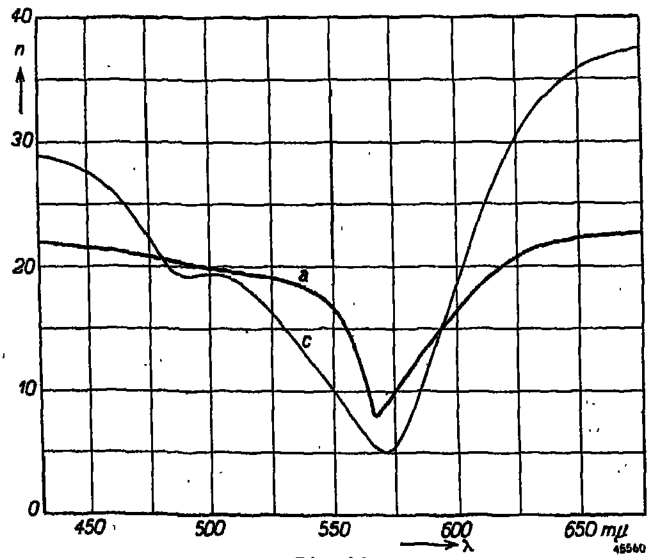


Fig. 98
The number of steps of just noticeable colour difference along straight lines connecting the white point with different spectral colours.: *a* Experimental values. *c*: see sect. 81.

varied curves in colour space, but it is difficult to get a general view of the material thus obtained. Early attempts, therefore, were made to obtain a systematic survey of this whole field. A formula or a precept was sought by which it would be possible to calculate the limens (and therefore the sensitivity of the eye to colour differences) along any given curve in colour space.

It is immediately clear, however, that such a systematic classification of the discrimination steps would give rise to far greater difficulties than the classification of the colour sensations themselves, such as was possible in the previous chapters. The cause of this difference is twofold: in the first place there are no such simple and strict laws for limens as was the case for colour sensations (Grassmann's laws), and in the second place the accessory circumstances under which the measurements take place influence the measurements to a much greater degree than colour measurements.

It appears from this last fact that if we should succeed in finding such a general precept, the possibilities of applying it would, for the time being, remain restricted to the conditions prevailing during the experimental researches. Through losing sight of this fact many investigators tried for a long time to find a formula that would hold generally; such attempts were of course doomed to failure. The earliest effort in this direction was made by Helmholtz (1891); it was based on the idea that Weber's law (sect. 77) would hold not only for the brightness of a colour sensation but also for the three basic processes by which Helmholtz supposed colour vision to be governed. This first attempt must be considered as a failure. In order to make his theoretical precept correspond with the experimental facts Helmholtz was obliged to make fairly arbitrary assumptions which were in conflict with his own general theory of colour vision (cf. sections 65 and 34).

A second interesting attempt was undertaken by Schrödinger[3]) (1920). Starting from a small number of assumptions, he constructed a theory — in an extremely elegant manner, mathematically — by which it would be possible to predict not only the sensitivities to all sorts of colour changes but also the results of a number of other measurements (among others those mentioned under B and C of sect. 76). Unfortunately it has since appeared from continued experimental investigations that almost all the assumptions from which Schrödinger started were incorrect or only correct in certain restricted circumstances. It is therefore not surprising that the numerous discrimination measurements carried out in later

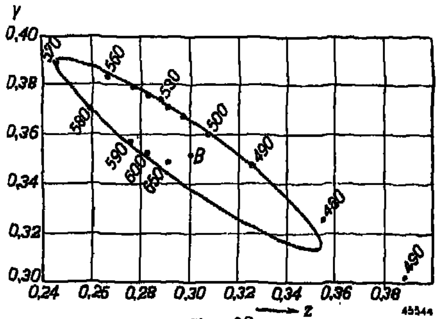


Fig. 99
The colours which are just distinguishable from the white B plotted in the $y,-z$ diagram.

years appear to correspond so badly with the formulae deduced by Schrödinger that we must also consider this attempt,' on a grand scale, to be a failure.

Subsequent investigators were more modest. In the first place luminosity variations were abandoned and all attention was therefore concentrated on differences measured under constant brightness (in this category belong all the investigations described in sections 78 and 79). Since by this restriction it was possible to illustrate all the results in a colour plane, the question automatically arose whether perhaps the size of the various steps was not bound up in a simple manner with the geometrical properties of the colour triangle. The simplest connection that can be imagined is that two equal distances in the colour plane correspond with the same number of steps of just distinguishable colour difference. If this simple connection existed it would follow that if a certain colour (for instance the white point) was plotted surrounded by all colours just distinguishable from it, these colours would then all lie in a circle with the chosen colour in the centre, for they are all one step removed from this colour.

We can now easily check by means of experiments if this is the case. In sect. 79 we discussed the threshold found by moving away from the white point in different directions: the limens for saturation differences in the neighbourhood of $p = 0$ (fig. 96). With these results we can calculate the locus of colours just distinguishable from the white point.

Fig. 99 shows the results of such a calculation (for Wright's results see curve b of fig. 96), on a greatly enlarged scale, of a part of the colour triangle. As we see, the points do not at all lie on a circle about the white point B: For the colour triangle, derived from the XYZ system prescribed by the C.I.E., the very desirable property that "all steps of just distinguishable colour difference are represented by equal distances" in no way holds.

§ 81 *The U.C.S. triangles of Judd and his school*

The negative result just obtained was in no way a discouragement for it brought to mind that there is actually an enormous number of possible colour triangles (see sect. 29, 30), all of which may be used to define colours unambiguously (fig. 16 shows one which differs greatly from the XYZ triangle). As the XYZ triangle is an arbitrarily chosen example from this collection, it is obvious that we should investigate whether perhaps some other choice of triangle would satisfy our purposes better, in which the various limens would be represented by approximately equal distances. It is already apparent from fig. 99 that this demand cannot be strictly fulfilled. The points plotted there mostly lie on the ellipse drawn, but at the two extremities (especially on the blue side) there are deviations. Now it is possible, by a simple transformation of the colour triangle, to change the ellipse into the required circle, but the deviating extremities will remain.

It must be said that with this renewed attempt quite a different course has been adopted from that of, for instance, Helmholtz[2]) and Schrödinger[3]). Where these investigators proceeded from certain theoretical considerations we have cast away all theory and set ourselves the purely practical task of representing the colours numerically in such a way that these numbers not only represent colours in a simple way but also the limens.

If the new colour space and colour triangle are to retain the same simple properties which apply in the XYZ system, the new coordinates must be simply related to the old. The methods of calculating the new coordinates from the old ones (transformation equations) must therefore be similar in form to equations (2) of section 30 which connected the XYZ system (chapter V) and the $B_1B_2B_3$ system (chapter IV). In passing from the $B_1B_2B_3$ system to the XYZ system the most simple properties remained valid.

Clearly the whole matter reduces to this, that a projective transformation of the colour plane must be found; it is clear that in this way the ellipse of fig. 90 can be transformed into a circle.

Judd[9]) was the first to follow this path successfully. He gave the following transformation equations:

$$\begin{aligned} R'' &= 3.1956\,X + 2.4478\,Y - 0.1434\,Z \\ G'' &= -2.5455\,X + 7.0492\,Y + 0.9963\,Z \\ B'' &= \phantom{-3.1956\,X + 2.4478\,Y + {}} 1.0000\,Z \end{aligned}$$

which therefore enable us to calculate the new coordinates

$(R''G''B'')$*) of any colour for which the X, Y and Z are given.

Fig. 100 gives the colour triangle (in the equilateral shape) which follows from the space coordinates R'', G'' and B'' $[r'' = R'' : (R'' + G'' + B'')$ etc.] with the curve of the spectral colours, the white point E, the line of the purples, that of the total radiators etc. In this triangle lines of equal length therefore represent, approximately, colour differences which can be bridged with an equal number of steps of just noticeable colour differences.

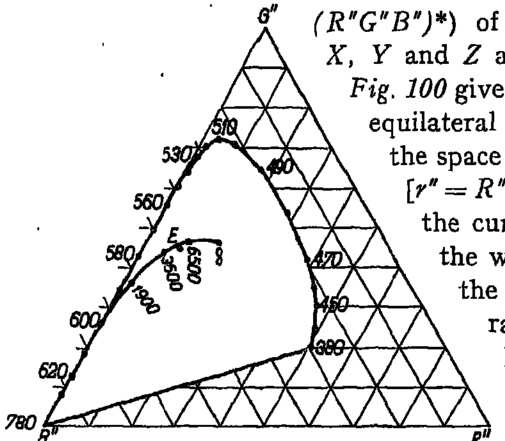


Fig. 100

Judd's U.S.C. triangle. Here the discrimination steps are approximately represented by equal distances.

Judd had various imitators. Thus McAdam[2]) gave other transformation equations with the same object while preserving more favourable properties of the C.I.E. system than was originally the case with Judd. Further advantages of this system are that the equilateral triangle is replaced by a rectangular one and the formulae are simpler; the new space coordinates (U, V, W) follow from:

$$U = {}^2/_3\, X, \quad V = Y, \quad W = -\tfrac{1}{2} X + \tfrac{3}{2} Y + \tfrac{1}{2} Z$$

Finally Breckenridge[1]) and Schaub have devised a new system (also rectangular) that is slightly more comprehensible than the other systems but otherwise lacks several of the advantages of that of McAdam. The equations which the space coordinates $(X'Y'Z')$ of this system give are:

$$\begin{aligned} X' &= && 0.82303\, Z \\ Y' &= -2.3280\, X + 6.4461\, Y &&+ 1.3690\, Z \\ Z' &= 2.9683\, X + 2.2475\, Y &&- 0.55176\, Z \end{aligned}$$

From these the trichromatic coefficients follow in the well-known manner: $x' = X' : (X' + Y' + Z')$ etc. If we now plot in a rectangular coordinate system the quantities $x'' = 0.075 - x'$ and $y'' = y' - 0.5$ an arrangement of the colours as in *fig. 101* is produced in which E is the white point: $x'' = y'' = 0$, while the four parts into which

*) The coordinates $R''\,G''\,B''$ are not to be confounded with those of sections 35 and 65.

the x'' and y'' axis divide the colour surface correspond approximately to green, blue, purple, yellow-red.

Finally the connection between the brightness of a colour sensation and its coordinates in the different systems must be mentioned:

$$\text{Brightness} = Y = 0.08852\ \bar{R}'' + 0.11112\ G'' - 0.09802\ B'' = 0.13857\ X' - 0.12182\ Y' - 0.09554\ Z'.$$

In America the systems dealt with in this section are combined under the name of "uniform chromaticity scale systems" (U.C.S. systems), which means that in these systems the colours as judged by the eye are equally distributed over the colour surface.

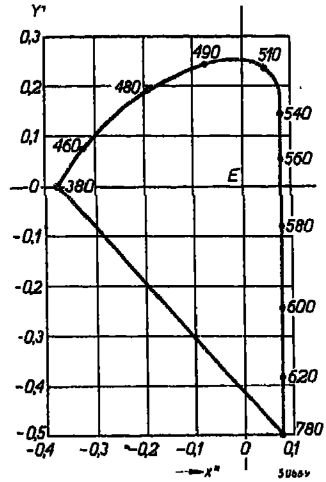


Fig. 101
The U.C.S. chromaticity diagram, according to Breckenridge.

Now to what extent do these systems fulfil the demands as regards limens? Judd worked this out extensively for his system. He was, however, obliged to use experimental material of very devious origins, that is to say, discrimination measurements carried out under very different conditions. There is a better control if we compare the limen predicted from a U.C.S. system with measurements carried out as far as possible by one and the same observer and all under the same conditions. We are enabled to do this by the experimental work of Wright [5, 6]) and his collaborators Pitt and Nelson, from which, among others, curves b of figs. 95, 96 and 97 have been derived.

Now we have drawn curves marked c in the figures 95-97, as predicted from the U.C.S. system of Breckenridge (fig. 101) if it is assumed that each limen in this colour surface is represented by a distance 0.0128.

The value 0.0128 is a compromise between the values required to adapt the empirical curves as well as possible to the experimental results of fig. 95 and 96.

From fig. 95 (curve a) we deduce that the spectrum between 440 and 630 mμ contains just 113 distinguishable colours. If we make the same demand of Breckenridge's system we must represent the limen by a distance of 0.0102. The best adaptation to the measurements of fig. 96 is found, however, by making this distance equal to 0.0160 (fig. 102).

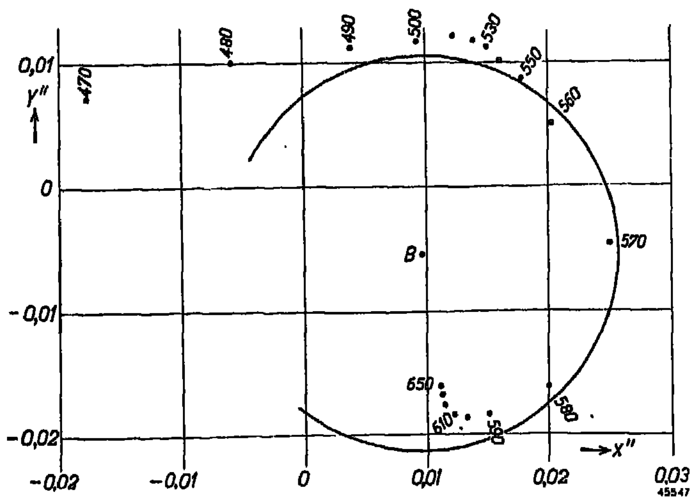


Fig. 102
The colours which can just be distinguished from the white B are plotted as in fig. 99, but this time according to Breckenridge's diagram.

Finally we have illustrated in *fig. 102* how the points drawn in fig. 99 lie in Breckenridge's $x''y''$ space. If we look at the figures 95-97 and 102 giving the comparison between Wright's experimental results and the predictions from the U.C.S. system, we see that the correspondence is still far from perfect.

It appears from fig. 102 that as much as can be attainable has been attained here: the points lying on the ellipse in fig. 99 now lie on a circle with a radius 0.016. In agreement with this, curve *c* in fig. 96 has also the same course as curve *b* over the greater part of the spectrum.

In fig. 95 the agreement is much worse; the maximum at 520 mμ particularly is too high and moreover is displaced. Curves *c* and *b* correspond completely within a factor 2. The worst agreement is found in fig. 97: in the blue as well as in the yellow (570 mμ) experiment and prediction differ by more than a factor 5. The sharp maximum at predicted 570 mμ does not occur in the experiments. Finally in fig. 98 a fairly reasonable correspondence between experiment and theory is found.

We shall summarize the results as follows:

a. In the neighbourhood of the white point and also for the unsaturated colours the U.C.S. systems give by their distances in the colour plane a fairly correct measure of the number of steps of just noticeable colour differences.

b. In the neighbourhood of the spectral locus and in general for very saturated colours, the agreement between the U.C.S. system and the experimentally determined limens is much worse.
c. As far as thresholds for colorimetric purity are concerned the U.C.S. system is useless.

If these three points are taken into account the U.C.S. system can still prove serviceable. It gives a much better impression of small colour differences (as judged by the eye) than the C.I.E. system. From recent measurements [(Wright[9]), McAdam[8])], Wright[10]) draws the following conclusion. If we set out in the C.I.E. triangle all measured limens (that is the various values of PP′ from fig. 93) a variation in length occurs, amounting in some cases to 1 : 20. If we transfer them to Breckenridge's system the maximum variation is reduced to 1 : 4.

A further improvement of the U.C.S. systems is not possible in a simple way: the best possible use has already been made of the four parameters we had at our disposal to give the colour domain the desired form (see sect. 34). The only way to achieve any improvement is to invent a more complicated precept by which we can have more parameters to hand in order to obtain a good fit with the experimental data. An example in this direction is van Sinden's attempt to invent a $(x_1x_2x_3)$ system by projective transformation of the C.I.E. triangle, in which for each limen the quantity

$$\left(\Delta \log \frac{x_1}{x_2}\right)^2 + \left(\Delta \log \frac{x_2}{x_3}\right)^2 + \left(\Delta \log \frac{x_3}{x_1}\right)^2 \tag{29}$$

assumes the same value as would hold for the quantity $(x'')^2 + (y'')^2$ with Breckenridge. A close examination shows that in this case we have not four but six parameters at our service and that the U.C.S. triangles are all special limiting cases of Sinden's general assumption. It is therefore not surprising that Sinden arrives at a slightly better adaptation to the experiments (especially for the left-hand side of fig. 95 and the right-hand side of fig. 98). It does not, however, seem desirable to us for the sake of this slight improvement to pass over to the very unsurveyable precept (29). We can consider such precepts as purely empirical attempts; we can also, like Helmholtz[2]) (from whom (29) was derived by Sinden), Schrödinger[3]) and Sinden[1,2]), represent them as conclusions of certain theories. Considered from this point of view Sinden's precept is not very successful, as he had to admit himself that in his "theoretical" derivation he had to make use of an assumption by Helmholtz which is undoubtedly incorrect.

A still more complicated precept is given by Silberstein[1]). In his speculations the fundamental assumptions are so concealed under the mathematical brushwood that it is difficult to make out whether his considerations have any theoretical value. They have certainly no practical significance. See further Silberstein[2,3,4]), McAdam[9,10]), Moon[2]), Stiles[4]), Wundheiler[1]).

CHAPTER XIII

A closer study of the character of colour sensations

§ 82 *The problem of adjustments to "partial equality"*

We now turn to those problems in which the eye has to fulfil the task described in section 76 under B. Of two colours in the field of vision one is kept constant while the other is altered along a particular curve (or straight line) in colour space until both colours correspond in a certain respect, for instance until they display the same apparent brightness or correspond in saturation. If we remember the contrast set out at the end of section 22, between the physically defined concepts of dominant wavelenth (λ_d), colorimetric purity (p), brightness (B) on the one hand and the intuitive psychological concepts of hue, saturation and apparent brightness on the other side, it is at once obvious that for adjustments to "partial equality" (equality in one particular attribute) we are continually on the (oft mentioned) border between the different sciences on which colour study rests. Thus the aim of heterochromatic photometry (sect. 11), for instance, is to form a judgment regarding the psychological property "apparent brightness" while keeping the physical properties λ and p constant. Such a mixed character is shown by practically all tests in which an adjustment to partial equality takes place.

Consequences of this are that our knowledge of this kind of adjustment is still only fragmentary, the results of different investigators often contradict each other (a result of the great influence of the attendant circumstances) and opinions as to the nature of the whole problem are widely divergent. On this last point, the investigators can be divided into two main groups.

The first group believes that the concepts hue, saturation and apparent brightness — which together characterize the colour sensation — are to a certain extent independent properties of the colour sensation, and that by arranging the tests suitably each of these properties can be studied separately. As an example heterochromatic

photometry may be mentioned, in which an attempt is made to study the apparent brightness separately and to ignore the differences in hue and saturation deliberately.

The other group (of whom Schrödinger was one of the principals) conceived the problem quite differently. The three psychological properties mentioned are, in their opinion, so closely interconnected that it would not be possible to isolate, as it were, one of them. Any attempt in that direction — therefore any adjustment to partial equality — is taken by them as an adjustment to maximum "similarity" between the two colour sensations. In other words, the point at which the two colour sensations display most similarity must be looked for on the curve along which the variable colour moves, no matter whether this similarity is expressed in the approximate agreement of one of the three properties mentioned or whether the two colour sensations resemble each other most in another sense. [Helmholtz [2]) already expresses this view.]

If the difference in view-point is formulated, the solution of the controversy must be sought, for the greater part, on psychological or even philosophical grounds, and it is well-known that, particularly in the latter field, contradictory opinions can remain side by side for centuries.

Schrödinger [3]), however, completed his views on maximum similarity with the assumption that the measure of similarity between two colour sensations is determined by the number of steps of just distinguishable colour difference. This would reduce the problem of "adjustment to partial equality" to a problem of limens. We should therefore adjust to that point of the curve on which the variable colour moves which — measured in limens — is situated closest to the colour point of the fixed colour sensation.

By this addition the experimental verification of the views of the second group becomes possible in principle; for, after an adjustment to partial equality, one can ascertain by discrimination measurements whether the adjustment is in agreement with Schrödinger's assumption.

It is important to choose the conditions with care, for in most cases the accuracy of such adjustments will be too small for a decision to be possible. For instance the variable colour sensation might move along a straight line (l) with a point A at which the brightness corresponds with that of the fixed colour sensation and a second point B at which the dominant wavelength is the same as that of the fixed colour K; we should then have to request an unprejudiced observer

to adjust to maximum similarity, to equal apparent brightness and to equal hue in succession. After this he must determine the distance from K to the different points of (l) in steps of just distinguishable colour differences.

The author's opinion on this question is that neither of the two extreme views outlined above is quite correct We can, indeed, in given circumstances, deliberately adjust to equal apparent brightness, hue or saturation (after overcoming an initial reluctance and lack of self-confidence we soon learn to make fairly reproducible adjustments), but the result is to such a high degree dependent on the conditions of the test that we cannot say that a certain apparent brightness, hue or apparent saturation indisputably belongs to a given colour sensation. Among the influential attendant circumstances we must undoubtedly also include the psychological adjustment of the observer: the results may be influenced in a great measure by the nature of the explanation given to the observer before the adjustment by the person conducting the experiment. Indirectly the personal view of the latter may also make its influence felt.

In general it will also depend on the setting up of the experiment and on the conditions whether the result will agree better with the one or with the other point of view.

§ 83 *Saturation*

We have already discussed the adjustment to equal apparent brightness (in which λ_d and p are constant) in section 77. We now turn to the adjustment to equal saturation with constant λ_d and brightness B. In other words, the variable colour is moved in a plane of constant brightness along a straight line through the white point and one has to determine which example in the series of colours thus selected produces the same apparent saturation as a fixed comparison colour possessing the same brightness but another dominant wavelength. We see here the same principle that is met with in all adjustments to "partial equality" (sect. 76): the colour to be adjusted moves along a curve in colour space which does not pass through the colour point of the fixed colour. The most careful investigations of this type are those of Klughardt[1]) and Richter[2]). Since the results of these tests may, in various ways, teach us something about the relations between the individual concepts, we shall discuss them fairly extensively.

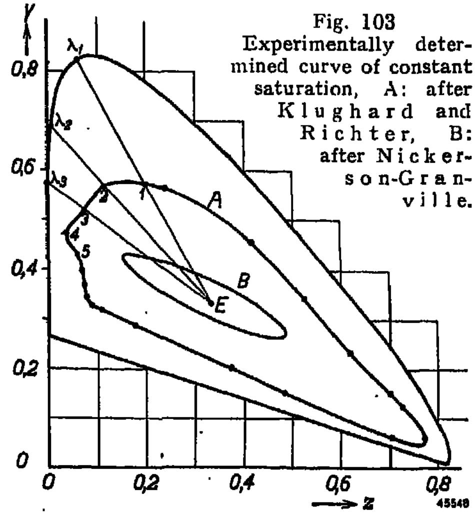


Fig. 103 Experimentally determined curve of constant saturation, A: after Klughard and Richter, B: after Nickerson-Granville.

Curve A in *fig. 103* shows the results. This curve was derived in the following manner. First of all a fixed colour, colour 1 (dominant wavelength λ_1), was chosen and on the line E λ_2 colour 2 was traced, which gave the same saturation as 1. Then 2 was used as a fixed colour and point 3 was determined on line Eλ_3 giving the same saturation as 2. Proceeding in this manner the curve was traced step by step: in 20 steps the starting point was reached, when the line joining E to the colours found (E 1, E 2 etc.) had turned through 360°.

We can therefore consider curve A as a curve of *constant saturation.* The method is strongly reminiscent of the step-by-step method mentioned in section 16 appertaining to photometry. In both cases small steps are chosen, as the adjustment would be too difficult and too inaccurate if large steps were made, i.e. if one attempted to compare two colours of a widely differing λ_d.

Both the psychological adjustment (the investigators had made it their purpose to find a curve of constant apparent saturation) and the method of measurement employed (thus the observer did not himself adjust by varying the colour sensation continuously, but received a number of samples in succession to judge) point strongly, in this investigation, in the direction of the views of the first group of section 82, so that we should do well to consider the results from that point of view too.

The opinion that in this experiment the property "apparent saturation" has successfully been isolated and measured in a certain sense, finds further support in the experimental fact that an actual "round" was made that ended in the same point 1, and that the direction of revolution had no influence on the shape of the curve A found. These are all properties that are self-evident if apparent saturation is considered as an independent unambiguous characteristic of the colour sensation; if, however, we consider the whole from the standpoint of adjustment to maximum "similarity" [according to Schrö-

dinger[3])] these properties ought then really to be called surprising.

If we take this last point of view the problem is then more or less analogous to the geometrical construction illustrated in *fig. 104*. By continually dropping perpendiculars we shall not return after one revolution to point 1, while, proceeding in the opposite direction, we shall not obtain the same curve! According to Schrödinger's later theory these differences would only lie in the fact that in determining the distance of two colour sensations we should have to make use of a non-euclidian measure; in that case the two experimentally found properties might still hold in special cases.

If we now try to bring curve A of fig. 103 into line with the other laws of colour vision it would be a matter of course to compare the curve of equal apparent saturation first of all with the formerly considered curves of equal colorimetric purity. Comparison of fig. 103 with figs. 31 and 32 shows that the "saturation" curves and the "purity" curves do not agree at all in form. Hence we can draw the important conclusion that neither the colorimetric purity p nor the excitation purity σ give a correct picture of saturation. If we move along a curve of constant colorimetric purity the saturation will vary. Conversely neither p nor σ possess constant values for the different points of curve A (fig. 103). This can be seen from fig. 105 where p and σ are plotted for the different points of the line of constant saturation (A) (as a function of dominant wavelength): on the left for the spectral colours, on the right for the purples (λ_d negative; see section 35). Since the curve of constant saturation does not therefore appear to be simply related with p or σ, we may now see whether the shape of curve A is connected with the discrimination measurements. Are the points of curve A perhaps all at an equal number of steps of just distinguishable colour difference?

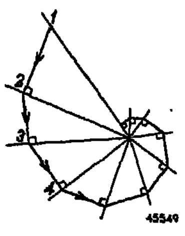


Fig. 104
From the point of view of an adjustment to maximum similarity the step-by-step method of fixing a curve of constant saturation is to be compared with the geometrical construction given.

In order to find this out we plot these points in Breckenridge's U.C.S. system (see fig. 101) and see whether the curve thus obtained is a circle (as was the case with the curve of fig. 99). The result is again disappointing: the curve marked UCS in fig. 105 gives the course of the distance of the various points from the white point, measured in Breckenridge's system (in arbitrary measure); this distance appears to be far from constant.

If we remember, however, that here we have applied the U.C.S. system to a case in which it is

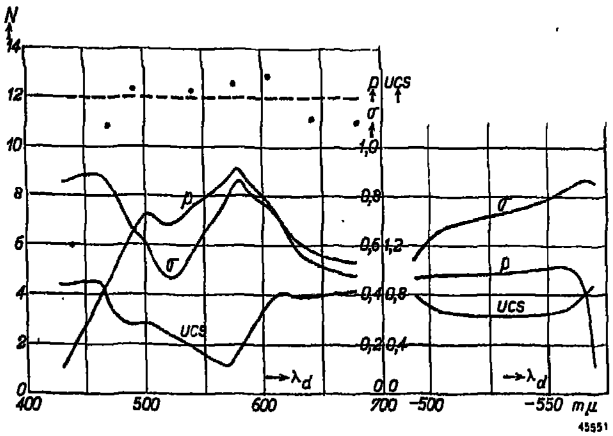


Fig. 105
Properties of curve A of fig. 103; from the curves p and σ it appears that neither the colorimetric nor the excitation purity is constant. The curve UCS gives the distance of the points of curve A to the centre in the UCS diagram and the points represent the same distances expressed in the number of steps N.

least useful (see the end of section 81: relatively high colorimetric purity; limens measured in the direction towards the white point) and if we notice that, as in fig. 97, the greatest deviations again lie in the neighbourhood of $\lambda = 570$ mμ, it would appear only right to compare curve A with some actual step measurements. The measurements of Jones[3]) and Lowry enable us to ascertain for 8 different wavelengths whether the distance of the points of curve A from the white point, measured in steps, is constant. In *fig. 105* the number of steps (N) is given for these 8 wavelengths. We see to our surprise that for 7 of the 8 wavelengths N is practically constant, namely 12 ± 1, so that (apart from the point at 440 mμ) we may conclude that under the conditions in which the measurements of Klughardt and Richter were carried out, two colours with the same saturation are an equal number of steps of just distinguishable colour differences from the white point. This thesis will undoubtedly not hold for all experimental conditions. Thus, for instance, for the points of a curve of constant saturation, which Nickerson[2]) and Granville measured under quite different conditions (curve B of fig. 103), the distance — measured in steps — to the white point itself is not even approximately constant. It will be interesting to fix the boundaries of the range in which this thesis does hold and to ascertain how the curves of constant saturation run when the eye has accommodated itself to strongly coloured light instead of daylight.

As for the deviating point at 440 mμ, we can only say that at this wavelength both the threshold measurement (influence of stray light, individual deviations of the relative luminosity curve) and the determination of the point on curve A (the close congestion of the curves of constant p) are very uncertain. In reality the thesis will probably hold here too.

It must be noted that curve A of fig. 103 coincides approximately over the greater part of its length with the curve of the C-colours (see fig. 59). This explains the fact that different members of the same colour circle in the Ostwald atlas (constant k, w and z) produce about the same apparent saturation.

§ 84 *Hue*

There are no complete and reliable data regarding adjustments to equal hue — in which brightness or colorimetric purity are kept constant. We do, however, know approximately how the lines of constant hue run in the chromaticity chart. They are more or less the straight lines joining the various spectral colours to the white point. In other words, the straight lines of constant hue coincide roughly with the straight lines of constant dominant wavelength (cf. for instance fig. 31). Here the question arises, however, which point in the colour triangle must be taken as the white point in this connection. This question is directly related to the phenomenon of chromatic adaptation of the eye, a phenomenon which we shall examine more closely in section 85. In general, it can be said that if we observe two small coloured light spots — when judging equality of hue — while the rest of the field of vision radiates mostly "white" light, we find that the lines of constant hue coincide approximately with the straight lines through the colour point representing the colour of the "white" used for the background.
It follows from this general rule, among other things, that the value of the dominant wavelength only gives a rough indication of the expected colour sensation when the monochromatic system (λ_d and p) is based on the white point appertaining to the environment (see also the example of fig. 25).
If on the other hand we observe the test colours against a completely dark background (as is usually the case with signals and with colour and discrimination measurements) we shall in general find different lines of constant hue. The results of various investigators are still very divergent on this point. Sometimes we find straight lines passing through a fixed point (there is no agreement as to the position of this fixed point), sometimes, on the other hand, we find systems of

straight lines which, when extended, do not pass through one point, and in certain circumstances even definitely curved lines [R o s e- m a n n [1])].

In any case it appears that the judging of equality of hue is very greatly influenced by all kinds of side issues. Among these the colour of the environment undoubtedly plays the chief part, but there are apparently still numbers of other factors. Especially in the neighbourhood of the curves of the spectral colours deviations from the straight line may occur [A b n e y [2]), J u d d [2])].

The judgment of hue differs from the judgment of apparent brightness and saturation in so far that, instead of comparing two colours, we can observe one colour separately and can always under all circumstances define the result by such verdicts as "the colour is blue, but has a slight tinge of green" etc. (the branch D mentioned in section 76). For the two other sensations this is only possible under certain circumstances and then in a very limited degree [see for instance J u d d [12])].

§ 85 *Chromatic adaptation of the eye*

In the course of this chapter, which deals more with colour sensations than with colours, we have spoken repeatedly of the influence of the colour of the environment and even from time to time used the expression "the eye is adapted to a certain environment colour". We shall now examine this latter phenomenon a little more deeply. In order to demonstrate the importance of adaptation we must mention a "law" that has aroused great interest for decades among psychologists and physiologists, namely the rule that the colour sensations produced by the coloured objects in our surroundings are approximately independent of the nature of the light by which they are illuminated.

Now this rule was laid down at a time when illuminants were almost exclusively those whose spectral distributions more or less corresponded with those of black bodies (fig. 37) (from candlelight to daylight). For lights whose colour points lie far outside this range or whose spectral distributions are quite different (sodium light, mercury light, neon light etc.) this rule does not hold at all (see, for instance, H e l s o n's [1]) work on strongly coloured illumination). Even with this limited validity the existence of this rule is really rather surprising. Let us now examine fig. 30, in which we calculated the colour in the C.I.E. system of the same purple-coloured paper

once by incandescent light (A) and once by daylight (C). The rule says that the paper should look approximately the same in both cases (when at least the whole background is illuminated by the particular lights). In other words, two colours (radiated by the same object under different circumstances) which according to fig. 30 differ very widely produce about the same colour sensation under the given circumstances. This pronouncement appears indeed to be correct.

How should such phenomena be explained? This problem again lies on the boundary line of various sciences, with the result that psychologists, physiologists and physicists who have occupied themselves in turn with it have mostly given far too one-sided explanations of the phenomenon and in most cases have not understood each other's lines of thought, far less appreciated them.

An almost grotesque example of the differences in meaning given by these groups to the same word is the word "transformation" used by both physicists and psychologists in this connection. The one understands by the term a mathematical calculation and the other a psychological process [see J a e n s c h [1]]. Some other literature on the various concepts is to be found in R i c h t e r [6]).

Only one conclusion can be drawn with certainty from the frequently contradictory speculations which have been made; and that is that the validity of the rule in practice is a consequence of the collaboration of some very divergent causes, some of which are as follows:

a. changes taking place in the retina under the influence of the colour of the surroundings in which we observe the colour (chromatic adaptation);
b. memory for colours and association. This means that we unconsciously associate a certain colour with a certain object. If we recognize a flower as a buttercup it requires a great deal of persuasion before we will believe that it does not look yellow;
c. various more complicated psychological phenomena of which the majority are combined by psychologists under the name of "transformations of colours".

The boundary between causes (a), (b) and (c) is not always sharp. At the risk of being accused of one-sidedness I shall restrict myself to the discussion of (a). This cause has been examined most thoroughly and explains the greater part of the phenomena. Moreover it is possible to study chromatic adaptation under such circumstances that the influence of both the other causes is largely eliminated.

Everyone is acquainted with the phenomenon of "adaptation to brightness", an automatic change in the sensitivity of the retina

which enables us to see reasonably well at extremely different levels of illumination, provided we have sufficient time, in the case of large sudden changes, to accommodate ourselves to the new circumstances.

Chromatic adaptation is a quite analogous phenomenon. Instead of the accommodation of the retina to the average brightness level in the field of vision, whereby a tenfold increase of brightness often hardly strikes us, here a partial accommodation of the sensitivities of the retina to the average colour occurs, whereby we hardly notice a fairly large change in the average colour of our environment. The ceiling of my room looks white both in daylight and by incandescent light (note, however, that in this example cause (b) may also play a part) [see Bouma [16])].

We have previously repeatedly ascertained that the brightness problem is one-dimensional, so that the brightness can be defined by the aid of one luminosity curve. The colour problem, however, is three-dimensional. For the definition of a colour it is necessary to take three luminosity curves into consideration. If we assume — in the spirit of von Helmholtz — that these three sensitivities belong to three independent processes taking place in the eye, then the supposition is obvious that the phenomenon of chromatic adaptation consists simply in a modification of the ratios of the three sensitivities. In this case the shape of each curve remains, of course, unchanged, just as in the case of the relative luminosity curve. We shall see in the following section what conclusions can be drawn from this simple assumption.

§ 86 *Chromatic adaptation; conclusions*

In the first place we shall consider the case of subjective colorimetry. We have made two light spots indistinguishable to the eye, while the spectral distributions of the two halves may still be considerably different. We conclude from the equality of the colour sensation that the three processes are stimulated as strongly by the one light mixture as by the other.

If we assume that chromatic adaptation consists of a change of the three sensitivities, that adaptation will therefore exert exactly the same influence on the appearance of one spot as on the other, so that, when the eye is accommodated to another colour the equality of the two colour sensations still remains. This property, known of old under the name of "the permanence of colour equations", actually

holds within wide limits. Hence in measuring colours we need never bother about the phenomenon of chromatic adaptation. This adaptation may alter the character of the colour sensations but cannot disturb their equality, and it is with this equality that colour measurements are ultimately concerned.

It should be noted that a similar law does not hold for heterochromatic photometry. If a heterochromatic adjustment to brightness has been made this adjustment can be disturbed if the sensitivity of the retina has been noticeably altered (for instance by fatiguing with coloured light, the introduction of a coloured environment, continued staring at the coloured field of vision etc.). This fact forms one of the causes of the uncertainty of heterochromatic adjustments (see sect. 11).

The phenomenon that a white surface — filling the greater part of our field of vision — remains white even if we change the colour of the light within moderate limits, also tallies with our simple idea regarding chromatic adaptation. In passing from one light to the other the colour coordinates of the white surface will change, the three processes will be stimulated in another ratio and initially the colour sensation will therefore also be changed. Then, however, chromatic adaptation takes place and it is clear that through a modification of the sensitivities the modification of the stimuli can be compensated, so that, finally, the signal to the brain will again become the same as before the change of illumination. But then we also get the same white colour sensation as before. If, however, we modify the illuminant too radically (for instance, by passing over to a saturated red), the chromatic adaptation can no longer keep pace. The change in stimulus is only partially compensated by a change in sensitivity and the result is that the white surface does not entirely regain the white colour sensation but, in this example, still gives a desaturated red sensation.

The phenomena described are quite analogous to the effects occurring when the brightness is modified. At first the apparent brightness will appear to us greatly modified, but adaptation will endeavour to compensate for the change of stimulus. If the change in brightness is not too great this compensation will be complete and we obtain the same apparent brightness as before the change; if the change is too great only a partial compensation will take place and the apparent brightness will change too.

The fact we have just described, that a large white surface still makes a "white" impression even if we modify the illuminant to a fairly great extent, is apparently a special case of the rule mentioned in section 85. How does this rule obtain in general? In other words,

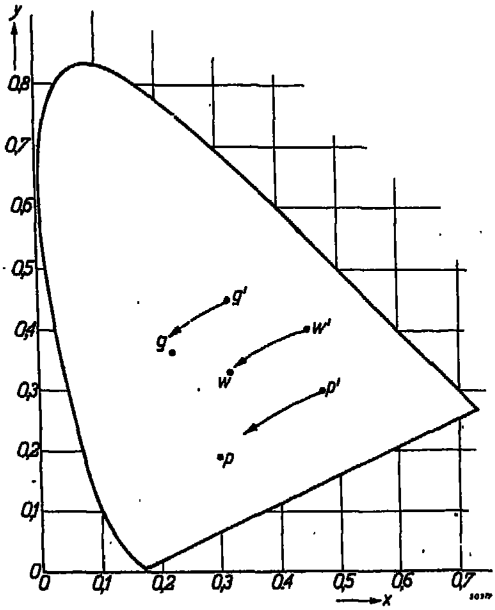


Fig. 106
Influence of the chromatic adaption on the colour sensation of a large white surface (w) and small coloured patches (g) and (p) present thereon. When switching over from daylight to incandescent electric light the chromaticity shifts to w′ g′ p′ but this is wholly or partly compensated by chromatic adaption, as indicated by the arrows.

is chromatic adaptation able to bring about a similar compensation for coloured objects in the field of vision at the same time as the compensation of the modified white surface?

If we again start from our simple assumption about the nature of chromatic adaptation, the answer to the latter question will in general be negative. For the modified ratio of the sensitivities of the three processes was already completely determined by the rule "white remains white", and it is unthinkable that the same modification in the ratios would give back exactly their previous appearance to the colours of all our surrounding objects as well.

That the compensation for colours must be incomplete in certain cases becomes clear to us if we observe, for instance, two coloured objects which have a considerably different reflection curve but produce the same colour sensation in daylight. If we now replace daylight by incandescent light that equality of colour sensation will, as a rule, be initially disturbed (this can be shown by calculation of the colour coordinates in both cases). The subsequent chromatic adaptation can never restore that equality (rule of the permanence of colour equations!) so that at least one of the objects does not return to its former colour sensation.

In *fig. 106* the colour points w, p and g are those assumed in daylight by a large white surface and two purple and green papers placed on it, while w′, p′ and g′ represent their colour points under incandescent light.

If we first examine the whole by daylight and later on by incandescent light we know already (owing to chromatic adaptation) that the

colour w′ will in the second case make the same colour sensation as the colour w in the first case (white remains white). We have symbolized that equality of sensations in fig. 106 by an arrow connecting w to w′.

Analogous arrows may now serve to represent the influence of chromatic adaptation on the colour sensation of the coloured objects present in the field of vision (examples g′ and p′). The beginning of the arrow always indicates the colour point that the object produces by incandescent light. The same sensation is produced as before the modification of the light species by the colour indicated by the point of the arrow. It is clear that a complete compensation would only occur if the point of the arrow reached point p (or g).

If we introduce the above modification of the light species with various small coloured objects on the white background, each coloured object will have its own arrow. The points of the arrows will all be different because the colour points of the objects, calculated for the illumination with the new light species, are all different. How can we now obtain a general view of the position of all such arrows starting from different points of the colour surface?

Since we started from such a simple notion about the nature of chromatic adaptation, there must be a simple connection between the coordinates (xyz) of any given colour at the foot of the arrow and the coordinates $(x'y'z')$ of the corresponding arrow head. It can indeed be proved that the connection between (xyz) and $(x'y'z')$ is of exactly the same character as the connection between two different systems of colour coordinates, for instance (rgb) and $(b_1\ b_2\ b_3)$ (see chapter V). Such a connection is called by the mathematician a projective connection.

The case mentioned above of two colours indistinguishable by daylight can be illustrated by *fig. 107*. In daylight the two objects produce the point k, and by incandescent light the two different colour points k_1 and k_2. The influence of chromatic adaptation is indicated by the two arrows, whose points can never coincide.

If we now keep to the colours most frequently occurring in nature, namely those displaying a more or less regular reflection curve and which are not too highly saturated, the displacements of the colour points (as w → w′, g → g′ and p → p′ in fig. 106) all appear to take place in about the

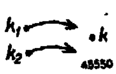


Fig. 107 When two objects (with different spectral reflection curve) have the same colour point k when illuminated with daylight, generally, on switching over to incandescent light two colour points k_1 k_2 will occur; this difference cannot be compensated by chromatic adaptation.

same direction, while the arrows have about the opposite direction and lead us back, approximately, to the starting point.
Under the same restricting circumstances the rule given at the beginning of section 85 therefore holds to a fairly close approximation. With the use of lights with widely differing colour points or spectral distributions, and with colours not answering to these limitations, the rule does not hold at all. Chromatic adaptation is not able to compensate for the colour difference. All these conclusions are in complete agreement with what is shown by experience.

In order to test the preceding considerations quantitively, Kruithof[3]) and Bouma[11, 16]) observed about a hundred different coloured cards of the Ostwald[1]) colour atlas (series nc) on a large white background, the objects being illuminated in succession by eight different light sources. The hue of each was decided upon, the observer having the choice between 36 colour names. In working out the results, besides the simple notion of the nature of chromatic adaptation, the approximation mentioned in section 84 for the lines of constant hue (coloured spots on large white background) was also assumed. Mathematically the following conclusions were drawn from the theory:

1. The lines of constant hue drawn for two lights give rise to two projective bundles. If the geometrical position of the points of intersection of pairs of corresponding rays (which therefore have the same hue) is determined, we find a conic section (hyperbola).
2. All the conic sections originating in this manner must pass through three fixed points.
3. From the position of these points we can deduce the sensitivities of the three separate processes in the eye.

All these theoretical conclusions are confirmed by experiments to a close approximation.
Other interesting contributions to this field are to be found in Wright[4]), Judd[12]) and Frieser[2]).

§ 87 *The estimation of colour differences*

The human visual organ is only to a certain degree able to compare differences of colour sensation (task C of section 76). This task can assume the following forms, amongst others (see also Boring):

a. Four colours A, B, C and D are displayed simultaneously. The observer must judge whether the difference between the sensations C and D is greater or smaller than the difference between that of A and B.
b. One of the four colours (for instance D) can be changed in a certain manner. The observer must make the difference between C and D equal to that between A and B.
c. In cases (a) and (b) the same colour can be chosen for both B and C; then only three colours are displayed and it must be judged

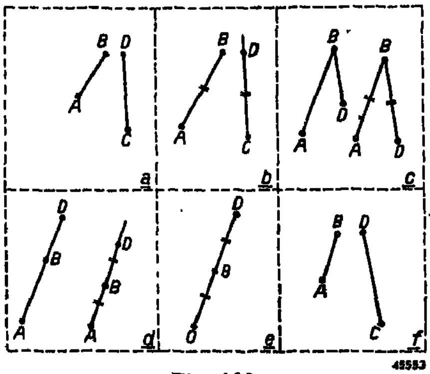


Fig. 108
Different methods of estimating colour differences. In the diagrams two differences which are estimated as being equal are indicated by equal distances.

whether A or D differs most from B, or an adjustment must be made in which both differences are judged to be equally great.

d. A special case of (c). B can be moved in colour space along a line on which A and D are also situated; decide whether B lies nearer to A or to D; adjustment until the differences are equal (therefore where — in a psychological sense — B bisects the distance AD) or an adjustment until B divides the distance between A and D, for instance, in the ratio 1 : 2.

e. A special case of (d). Colour A has brightness 0. Two colours only are therefore displayed and the task is to adjust colour B in such a way that the apparent brightness of B is half as large, for instance, as that of D,

f. For the four fixed colours, A, B, C and D the ratio of the difference CD to that of A and B can be estimated [B a l i n k i n [1])].

Fig. 108 shows the possibilities a-f schematically. For differences judged "equal", equal distances are drawn. The estimation of colour differences illustrated here has hitherto been more or less exclusively applied by psychologists. The more physically minded investigators were usually reluctant to carry out such experiments. The psychological nature of these adjustments was too great for their taste, the accuracy too low and they were not able to refute such remarks as "it is really nonsense to speak of one apparent brightness being half as great as another" clearly and in the language of physicists.

Added to this was the fact that they thought they possessed a general method which would make such estimations superfluous, namely the measurement of steps. The difference of the sensation given by colours A and B is expressed in the number of steps of just distinguishable colour difference which must be made in colour space in order to pass from A to B by the shortest path (here the name of S c h r ö d i n g e r must be mentioned once more).

Various objections can be raised against this method. Thus it is not at all a matter of course that for large differences the colour differen-

ces estimated are actually proportional to the number of steps. Moreover, very often the number of steps of a U.C.S. triangle (see section 81) are used for the calculation. These triangles are based on measurements carried out under circumstances which differ entirely from the conditions under which it is desired in practice to apply the results. Very little is known about the seriousness of the latter objection. What happens, for instance, to the steps when chromatic adaptation of the eye takes place? In summing up we can say that measurements of the type a-f are wrongly avoided by a certain group of investigators; they are necessary to check the validity of the step method; on the whole there is more advantage to be had from a direct measurement — even if less accurate — of a magnitude we desire to know, than from an accurate measurement of a property which may not be relevant. As for the repugnance to apply such estimations it is, as a rule, an exactly similar case to that of adjustments to partial equality: at first most observers experience a slight reluctance to performing the task, but when this reluctance has been overcome one is surprised that the adjustments and estimations appear to be much more accurate and much more reproducible than one had thought.

Literature on the estimation of colour differences: Boring[1]), Newhall[1]), Richardson[1]), Helson [1,2]), Judd[12]), Balinkin[1]) etc. As regards analogous estimations in acoustics see Geiger[1]), Stevens[1]), Newman[1]), Richardson[2]) etc.

§ 88 *The naming of colours*

Closely related to the problem of the estimation of colour differences is that of the naming of colour sensations (task D of section 76). The difference is that here the attention is fixed on only one colour and one tries to define one's impression by a name. This is easiest for hues for which there are names in general use: red, orange yellow etc. Many authors show how the hue names are divided over the spectrum. Table 8 gives a survey of the average results of a very large number of investigators [see Judd[12])]. In the column "average" the average wavelength values (in mμ) are given representing the purest orange, yellow etc., as well as the λ at which the one colour passes over into another. In the column "Variation" the boundaries are listed within which the results of various authors are contained. No average value is given for red, as according the various authors the "purest" red does not occur in the spectrum but lies a little further on, on the purple line.

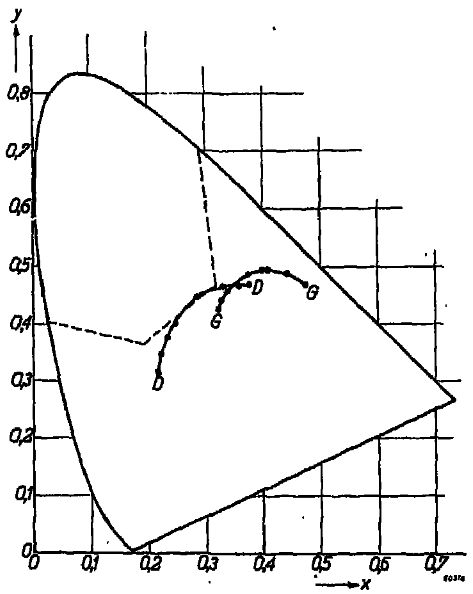


Fig. 109
The part of an Ostwald circle nc which, when seen against a large white background, is arranged under "green", DD with daylight, GG with incandescent electric light. As regards the dotted line see fig. 110.

The "variations" mentioned in table 8 are surprisingly great. Thus the average wavelength for pure yellow is 577 mμ, but that same wavelength is given by one author as the boundary between green and yellow, while the average value for the green-blue boundary (495 mμ) is called "pure green" by one of the investigators. These variations are not to be ascribed in the main to individual differences in the colour vision of the observers but to differences in the circumstances in which the observations were carried out.

The influence of the environment on the judgment of the hue of a coloured card comes clearly to the fore in *fig. 109* (from the investigations of Kruithof[2]) and Bouma[11]). Here the curve DD contains the colour points of the coloured cards of a complete Ostwald "colour circle" which were called green (or more accurately between yellow-green and greenish blue) when observed against a white background illuminated by daylight. The curve GG contains the colours called green when the large white background was illuminated by incandescent light. It is obvious here what a great influence the changed chromatic adaptation has on the judgment of hue.

The average values of table 8 are characteristic for the observation of fairly small spots of light against a dark background, a method of observation which is of special importance in studying colours for signalling purposes.

For the influence of the brightness of a colour sensation on the judgment of its hue (observed in a dark environment) see Ornstein[1]) and Purdy[2]).

The accurate definition of the spectral colours which make impressions of pure yellow, green or blue on the eye has also aroused much

interest in psychological and physiological circles, attempts having been made to relate these colours to the "primary colours" of the various theories on colour vision.

From the measurements of W e s t p h a l[1]), P u r d y[8]), D r e v e r[1]) and H u b b a r d[1]) follow the average values 574, 509 and 475 mμ (in fairly good agreement with table 8). T r e n d e l e n b u r g[8]) gives 570-585, 500-520 and 470-480 mμ. See also D i m m i c k[1]) and B e z o l d[1]).

Besides the previously mentioned property of these colours, that psychologically they give the impression of being pure and non-composite, these colours show a certain stability lacking in other colours. Thus their hue changes little or not at all when the brightness changes, or when white light is added or if the sensitivity of the retina changes.

The colour which psychologically gives an impression of being "pure red" does not occur in the spectrum, according to most psychologically interested investigators — as we have already remarked before — but is obtained by the addition of a little violet to the extreme red side of the spectrum (it therefore lies on the purple line in the colour triangle). Remarkably enough most physicists do not agree with this pronouncement.

TABLE 8

Colour	Colour-boundary	Aver-age	Variation
Red			
	red-or.	627	604—656
Orange		598	592—608
	or.-yellow	589	585—597
Yellow		577	568—585
	yellow-gr.	566	558—577
Green		512	495—535
	green-bl.	495	487—502
Blue		472	452—485
	bl.-violet	436	421—456

It is practically impossible to define apparent brightness and apparents saturation by names if the colour to be judged is the only one occurring in the field of vision. If the colour, however, appears against an extensive background, itself displaying some colour or other, it is possible to give the colours suitable names, for instance light blue, very pale green, etc. By a combination of such a name with the estimation of colour differences J u d d succeeded in defining numerically the apparent brightness and the saturation in a differently coloured environment.

In America great use is made at present of the colour atlas of M u n s e l l (see section 56) for naming colours under normal circumstances [J u d d[13]) and K e l l y]. In this atlas each colour sample is characterized by three magnitudes: value, hue and chroma, the first of which is defined as the brightness relative to a magnesium oxide surface illuminated by the same illuminant (standard white C), expressed, however, in a certain scale, so that for the colour white, for instance, equal steps in value correspond to equal increment of apparent brightness. The concepts "hue" and "chroma"

roughly equal our concepts of hue and saturation. Here too, however, the scale has been selected in such a way that equal increments in number correspond as far as possible to equal differences in hue and saturation. [An extensive survey is given by Newhall[2]), Nickerson[5]) and Judd.] In accordance with what has been said in section 84 the lines of constant "hue" in the C.I.E. chromaticity diagram are not straight and, moreover, depend on the value. The lines of constant "chroma" in no way coincide with those of constant p or σ (see sections 83 and 75a).

CHAPTER XIV

Practical applications

§ 89 *Illuminating engineering. General applications*

In this final chapter we shall try to present a survey of the branches of science and technology in which all that we have hitherto dealt with is put into practice. It is inevitable that we shall be compelled to treat most subjects only very briefly while other aspects can only be hinted at. For the field of practical application is so large that hundreds of pages would be required to give even a fairly detailed survey. We shall begin with the most obvious field of application, that of the science of illumination, and especially illumination for general purposes.

Let us consider the living room as an example. In most middle-class families reading, writing, sewing, recreation and the consumption of meals take place in the living room. Illumination must be such as to take all these occupations into consideration. The first three require a quantity of light falling on the work (illumination level), which must be suitable to the nature of the work. Further it is important how the light falls (avoidance of troublesome shadows and high lights, not too great local brightnesses or contrasts in the field of vision etc.). The two last mentioned occupations make quite different demands, in which colour plays a great part. Recreation, conversation and suchlike require in the first place a cosy, tranquil and aesthetically pleasing appearance, while moreover during the consumption of meals it is of great importance that the illuminated objects appear "natural". Should potatoes or butter assume a greenish tinge our appetite might be affected!

The requirements summed up here also apply to general illumination of another kind (offices, factories, shops, street lighting etc.) with of course the difference that sometimes one requirement and sometimes another is of more importance. In many cases the luminous efficiency of the light sources (the quantity of light received in return for a certain quantity of electric energy) plays a very important part in the choice of the kind of light.

Some decades ago problems of colour were fairly unimportant, for

the simple reason that there was still very little choice in colour and spectral distribution of the light sources. Most light sources had the character of temperature radiators (see sections 42 and 43) and were chiefly distinguished by different colour temperatures. Nor did this distinction form any stimulus for the development of colorimetry. For if we know that it is a temperature radiator we can determine the colour much more simply than with the aid of the more or less intricate means at the disposal of general colorimetry. The luminous efficiency, electrical properties, measurements of the ratio between the power radiated at two particular wavelengths, colour match with a light source whose colour temperature can be measurably regulated, these are the means by which we can obtain more or less accurate information regarding the colour temperature, which completely determines the colour properties of the light source.

The requirements of pleasant surroundings and natural appearance of colours were complied with fairly well. In fulfilling the last requirement chromatic adaptation (sect. 85, 86) of course plays a great part. The luminous efficiency of temperature radiators increases greatly with the colour temperature (sect. 42) but in spite of the great improvement in this respect brought about in the course of time (from 1900° to 2900°) the relatively low luminous efficiency was always the weak side of the light sources used. And all attempts to improve the colour properties (coloured lampshades, sunlight lamps) lower the luminous efficiency still further!

A new era dawned when gaseous discharge lamps made their debut in illuminating engineering, *i.e.* the sodium lamp and the mercury lamp [see for instance Uyterhoeven[1]), Oranje[1]), E. Dorgelo[1]), Cotton[1])]. The demands made by each branch of practical application forced themselves on the attention of scientists. Thus it appeared directly that both sodium and mercury light were unsuitable for the living room: sodium light on account of the impossibility of rendering colours and mercury light because colours appeared unnatural in this light.

When used for other purposes (for instance for street lighting) with quite different requirements, the new light sources were extensively employed. Sodium light especially, with its high luminous efficiency and its excellent qualities in connection with seeing on illuminated arterial roads, proved to be an ideal kind of light.

Attempts were soon made to extend the area of employment of mercury lamps. By two different methods, while preserving entirely or partially the high luminous efficiency of the mercury lamp,

attempts were made to improve its unpleasant colour properties. In the first, mercury light was combined with incandescent light giving rise to the so-called blended light. By sacrificing a part of the high luminous efficiency a better colour rendering for coloured objects was attained. Such combinations are frequently employed in factories and workshops and also in offices. In this last case, however, the quality of the colour rendering has already reached the limit. For the living room blended light is quite unsuitable. Here much higher demands are made as regards a natural rendering of colours. An important advantage of blended light is that with a well chosen mixture ratio its colour point approaches so close to that of daylight that during twilight daylight and blended light can supplement each other without unpleasant effects such as "false light" and coloured shadows etc.

The second method of improving mercury light is much more attractive as no reduction of the luminous efficiency need occur. Here the important quantity of energy emitted by the mercury lamp as ultra-violet radiation is transformed into visible light by the aid of fluorescent materials. By a suitable selection of these substances the mercury spectrum can be supplemented in the spectral bands where this is most needed (especially in the red). The first steps along this path did indeed lead to a considerable improvement, but the colours were not good enough for use in living rooms.

In this period of development when more and more possibilities of coloured light sources lay open to use, it was natural for interest to be aroused in methods for defining the colour of the light source in the colour plane [see B o u m a [7, 8]]. It was not long, however, before it was seen that the kind of light was not sufficiently characterized by its colour point; and in particular that it is not possible to deduce from the colour point how the colours of the surroundings will look by that light. An extreme example of this fact has already been seen in section 23 in the comparison of daylight with a light source showing the same colour point but consisting of only two complementary wavelengths. A practical example derived is that of a blended light possessing the same colour point as sunlight, but rendering certain colours very unnaturally.

In order to predict the colour rendering completely we must, apparently, know the complete spectral distribution of the light source. But even if this is known, it is still difficult, for instance, to compare fluorescent mercury light with daylight for it would be desirable to "spread out" the contribution of the mercury lines over a certain

spectral range, in order to obtain a spectral distribution practically equivalent to the original and more easily compared with a continuous distribution, such as that of daylight.

With this purpose in view Bouma [7, 8] proposed to divide the spectrum in the following 8 wavelength intervals:

$$400 - 420 - 440 - 460 - 510 - 560 - 610 - 660 - 720 \text{ m}\mu$$

and to determine the contribution to the light flux radiated in each of these bands. The choice of the 8 bands is such that two light sources whose light flux contributions in each of the 8 bands correspond will also differ as little as possible in all their other colour properties.

An apparatus has been constructed by van Alphen [1]) for measuring these eight contributions directly. Other attempts in this direction are those of Richter [4]) (who worked with four filters), Taylor [1]) (a division similar to the one mentioned above but with less happily chosen intervals), an early proposal of Ives [5]), and recently Harrison [1]) and Aldington [1]). Various methods have also been invented for the immediate estimation of the colour sensation, given by objects under a certain light source, for instance by Richter [4]) and Bouma [7, 8]). We would stress the fact that in all these methods the phenomenon of chromatic adaptation is not taken into account.

The problem of the different coloured light sources has entered into a new phase since the invention of a new light source. There are the tubular low pressure mercury lamps (such as the MCF/U lamps manufactured by Philips) with fluorescent material on the inside of the tube, which contain a considerably smaller proportion of the troublesome mercury lines, and transmit an important part of their radiation in the form of the continuous spectrum of the fluorescent material. By varying this material the most varied colours can be attained [Kruithof [1])], and the appearance of the coloured objects is with most of these types of lamp so satisfactory that they are suitable for the living room. While formerly it was sufficient to imitate daylight or incandescent light as well as possible by the aid of mercury lamps, the field of investigation has now been considerably extended in the sense that all kinds of different light species are being studied to see if they are suitable for interior illumination. The following points form the preliminary results of this investigation:

1. The colour point of the light source should not be too far removed from the black body locus (see fig. 38).

2. Deviations to the green side (in fig. 38 above the locus) are much sooner unpleasant than those on the other side.
3. The spectral distribution must be as continuous and even as possible.
4. If the colour point lies on the black body locus the appropriate colour temperature should preferably lie between 3000 and 5000°; as the illumination level is raised so, on the whole, a higher colour temperature is preferred [K r u i t h o f [1])].

Undoubtedly chromatic adaptation will have to be taken seriously into account with all further investigation into the most suitable spectral distributions. The study of this phenomenon is still very young.

§ 90 *Illuminating engineering. Special cases*

Having given a brief survey in section 89 of the part colours play in illumination problems of a general nature (illumination of living rooms, offices, factories, arterial roads etc.), we now wish to touch lightly on some special cases. In the first place there are various cases in which a light species is required to imitate daylight as closely as possible. We have in mind museums, and shops where coloured materials are sold. Formerly the only possibility was incandescent lamps, to the light of which — by means of filters or coloured bulbs — a spectral distribution had been given approaching that of daylight. The imitation was very imperfect and a great deal of light was lost. The closer the imitation of daylight the lower became the luminous efficiency of the whole. In these cases the fluorescent low pressure mercury lamps mentioned above will find an ever growing field of employment.
In the second place come those cases in which it is necessary to compare and judge colours as accurately as possible. For this, incandescent light is on the whole less suitable. As blue rays are only weakly represented in this type of light it is not possible accurately to distinguish colours whose reflection curves differ chiefly in this range. If it is required to distinguish the most divergent colours well, a light in which none of the colours of the spectrum is missing or too weakly represented must be chosen. Lights which correspond approximately to daylight fulfil this requirement best. If only a limited group of colours are to be clearly distinguishable (the sorting of particular products according to their colour, for instance, in the cigar industry) special spectral distributions can be indicated which

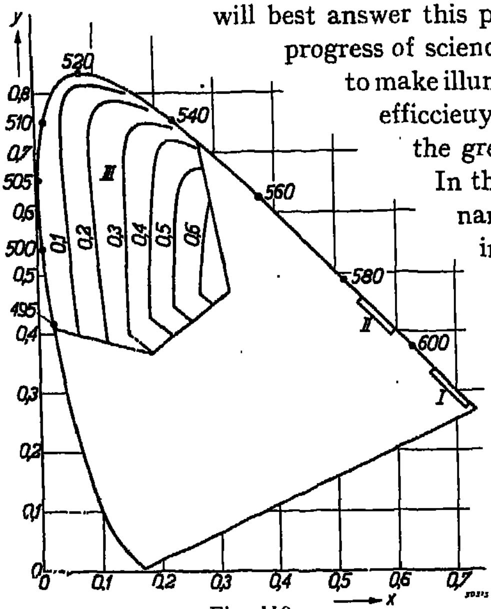


Fig. 110
The location of the aviation colours red (I), yellow (II) and green (III) in the colour plane. In the green area the maximum transmission factors are indicated (see also figs. 56/57).

will best answer this purpose. In this case too, the progress of science that enables us at present to make illuminants with a high luminous efficieuy in very varied colours, is of the greatest importance.

In the third place coloured illuminants are used for interior lighting for the purpose of creating a certain atmosphere or certain artistic effects (festive illumination, theatres). The range of colours suitable for these purposes is of course very extensive. To prevent unpleasant surprises illuminants should preferably be selected whose spectral distributions are more or less regular and continuous. In choosing illuminants in these cases, the material to be illuminated must of course be taken into account. Up to now we have only considered cases where a particular appearance must be given to the objects in our environment. We have seen that in such cases it is insufficient merely to give the colour coordinates of the illuminant; we must also have some idea of the course of the spectral distribution.

We shall now turn to cases in which it is not the intention to illuminate objects but to shine the light of the illuminants straight into the eye. In these cases spectral distribution plays hardly any part at all and the indication of the colour point is on the whole sufficient to characterize the light species. Beside the field of illuminated advertisements the use of illuminants in connection with signalling must be mentioned here. In the latter case, the idea is to send certain information with the aid of different coloured illuminants to persons at a distance (for instance, "safe", "unsafe", "you can land here", the course to be followed). It is of the utmost importance here that the person signalled to should be able to decide with certainty the colour of the signal observed. This decision consists in practice of an estimation of the hue. This is only possible with certainty if

a. the observer has a normal colour vision;
b. the colours are sufficiently saturated;
c. the colours do not lie in transition areas (for instance yellowish-green or greenish blue);
d. the recognition of hue is not too much influenced by any possible coloured environment (chromatic adaptation etc.).

These factors are taken into account in international agreements covering a number of public services (air traffic, road traffic and railways) regarding the colour range within which the different signal colours must lie. This range can be indicated in the chromaticity diagram. *Fig. 110* is an example, showing the ranges within which aviation red, yellow and green must lie (see also figs. 56 and 57). In the green range the maximum attainable transmission factors have been indicated (section 47). In fig. 109 (see section 88) the green range has been plotted for comparison with the hue estimations given there. From this we see that all colours in the prescribed green range are all properly saturated. The cards used for the measurement of the curves DD and GG had a fairly saturated colour (they belonged to the series nc, practically the most saturated series of the Ostwald colour atlas) and yet they lie outside the prescribed green range.

Further we notice that the position of the green range fits the curve DD much better than the curve GG. Hence we conclude that even in daylight the green range of fig. 110 is still serviceable, but that in incandescent light the prescribed range is not covered by the range of colours the observer will call green.

It is interesting to compare the limits for the very saturated colours of the green range with table 8 (sect. 88) in which we read that the spectral colours whose wavelengths lie between the limits 495 and 566 mμ are called green. The green range of fig. 109 extends on the blue side right up to the limit of 495 mμ while on the yellow side it does not extend anything like as far as 566 mμ. This is related to the fact that one must be careful on the yellow side in all circumstances to prevent a confusion with yellow signal lights. Since blue light is practically never used for signals, the danger of such confusion on the other side of the green range is not so imminent.

The ranges of the aviation colours can be described numerically as follows: red: $y < 0.335$; $z < 0.002$; yellow: $0.402 < y < 0.460$; $z < 0.007$; green: $z > 0.610 - 0.829\,y$; $z > 1.17 - 2\,y$; $0.76\,y > 0.173 + 0.24\,z$.

Extensive investigations on colours in connection with signalling have been carried out by Holmes[2,4]) and Hill[1]).

§ 91 *Application to trade, industry and science*

The science of colours is put to practical use in numbers of ways in trade and industry, in the first place as a method of defining the colours of the most divergent articles (dyes, enamels, paper, textile, flowers etc.). These are articles for which the colour is one of the most important properties. Two obvious methods exist for defining and afterwards reproducing these colours, namely by giving the trichromatic coordinates or by making and keeping samples for comparison. Both methods have their advantages and disadvantages. The indication of colour coordinates (for instance the x and y and the reflection factor when illuminated by one of the standard illuminants) has the advantage that it is easier to distribute three numbers among a great many interested people than actual colour samples. *Moreover in this way the danger of the sample modifying its colour* in any way in the course of time is also excluded. On the other hand it is true that the method of using coloured samples — especially for the layman — gives a good representation of the colour sensation to be expected far more directly. It must be remembered too that a relatively intricate measuring method must be applied to attain the accuracy possible by direct comparison with a colour sample.

These two objections have hitherto greatly restricted the field in which colour coordinates are applied. In certain cases we can check colours much more simply than with a trichromatic colorimeter. If, for instance a manufacturer of enamels wishes to ascertain whether the colour of his product remains constant during the course of time, it is sufficient for him to measure the spectral reflection factor for four wavelengths. An undesirable change of colour will then certainly appear in at least one of the factors [W e n t [1])]. Once a certain colour is fixed, the question at once arises in practice how great is the tolerance. How much may a colour deviate from the fixed colour and still be acceptable? It is self-evident that in determining such tolerance the sensitivity of the eye to colour differences, and therefore the size of the limens plays an important part. It can therefore be understood why it was for a long time thought possible to solve the problem of tolerance by the following simple rule: the colour may not deviate more than n times the just distinguishable colour difference from the fixed colour. Here the factor n depends, of course, on the nature of the application. A further simplification was brought about by the introduction of the U.C.S. systems (see section 81), in which the problem was thought to be

solved as follows: if the prescribed colour is marked in a U.C.S. system we can consider another colour as "sufficiently corresponding" if the colour point of that colour lies within a particular circle, the centre of which is the prescribed colour point. This precept was extended later on, in order to take differences in reflection factor into account [J u d d [1])].

The conviction has steadily grown, however, that the problem cannot be solved in practice in such a general manner. The acceptable deviations are determined jointly by a number of factors which have been left out of account in the above formulation. The same is true for the definition of a favourable spectral distribution of an illuminant (sect. 89).

Statements such as "a deviation of the colour point from the black body locus is most troublesome in the green direction" or "butter may look a little pale but on no account green", the simple step method cannot take into account. These and similar considerations have led to the view that it is necessary to determine the admissible deviations experimentally for each individual case.

The tolerances thus found can then be fixed in the colour plane, and afterwards it can be ascertained by a trichromatic measurement, whether a certain colour lies within these tolerances [M a c-A d a m [6])]. In addition it may also be desirable to indicate a tolerance for the reflection factor.

Summing up, we can say that the important problem of tolerances is far more complicated than would at first appear, and that it is far from being completely solved.

In America particularly great interest is shown in this problem, witness for instance the "Symposium on colour tolerance" (reprint from Am. Journ. of Psych. 52, 1939) in which the problem has been examined by a series of experts of the most divergent kinds [B o r i n g [1]), H e l s o n [2]), B a l i n k i n [1]) etc.].

The problems of colorimetry are of course of special interest to the dye industry. Several of the problems treated previously are gratefully put into practice in research work here. Let us quote as an example the theory of the optimal and ideal colours (section 47), which limit the satisfiable demands in the matter of high reflection factors and high colorimetric purity. The problems of dyes are complicated by the fact that besides additive colour mixing subtractive mixture often plays an important part.

See D o h n e r [1]), D u n c a n [1]), E l l i s [1]), H a r r i s o n [1]), G r u n d y [1]), V i c k e r s t a f f [1]), W h i t e [1]).

Besides the examples already mentioned there are also numerous cases in which the colour itself seems not to be of such great importance, except that it can give us indications of the quality, purity and other important properties of the products studied. As an example of cases where colour measurements are applied in this sense, the examination of sugar, various agricultural products, tobacco, oil, beer etc. may be mentioned.

See Moon[5, 8]), Reimann[1]), Taylor[3]).

Finally something must be said about the practical use in more scientific fields. Colour descriptions are regularly to be met with in scientific descriptions of natural objects (plants, animals, minerals). If these colours are to be accurately described, the proper way would again be the measurement and representation of the colour coordinates. The same is true of the various branches of chemistry. In pure physics there are fewer possibilities of practical use than one would be inclined to suppose. On the whole the physicist is more interested in the spectral composition than in the colour sensation that light produces. Yet there are cases where colour coordinates have their uses. A simple example has already been seen in the boundary colours (chapter VII). Much more complicated phenomena may appear when diffraction, double refraction, polarisation etc. play a part. In general it may be said that when we have an optical phenomenon entirely at our command we know the spectral distribution of the light that strikes the eye from any given point of the field of vision. But in that case it is also possible to calculate the colour coordinates of all points of the field of vision and so to describe completely the colour phenomenon as it appears to the eye. Such a calculation may be of service to us in two different respects. In the first place it enables us to describe the colour phenomenon accurately to other people and even, simply with the assistance of the calculated trichromatic data, to make a coloured reproduction of the phenomenon, which may be of importance for teaching purposes, for instance. In the second place the calculated colour points may serve to check our calculations and theories: by an attentive observation of the field of vision we can ascertain whether the image observed agrees with what was theoretically predicted.

As an example we might mention the colours of the rainbow. This well-known natural phenomenon conjures a play of colours before our eyes, which is approximately a prismatic spectrum. Popularly the spectral colours are indicated by the name of "colours of the rainbow". According to the classical theory of Descartes (1637) the rainbow colours are indeed practically pure spectral colours, determined by the dispersion

of drops of water in which the sunlight is refracted. A more accurate theory given by Airy and taking into account the diffraction offers a finer analysis. This theory also explains the repetition of colours one often sees on the inside of the inner bow. Perntner [see Prins [1]] on the basis of this theory described the colours of the rainbow in great detail. Prins [1] and Reesink have computed the colour points for different sizes of drops (500 and 50 μ) and sun diameters (0′ and 32′). The same authors also calculated the colour points of a number of diffraction patterns [Prins [2]].

§ 92 *Colour reproduction; additive method*

By the reproduction of colours we understand in general the making of pictures of objects in natural colours. This occurs in the first place in the art of painting, which we shall not discuss in this book, and also in the following technical processes, i.e. colour printing, colour photography, colour films and colour television. Much might be written about the various processes that are used in these arts. We shall therefore have to restrict ourselves to a few remarks of an essential nature showing the connections with the subjects already dealt with in earlier chapters. The principles are the same for all four cases: they are again founded on the number three that governs the properties of our colour vision and the whole of colorimetry. Further there is a close analogy with the "photo-electric-tricolorimeter" as used for colour measurements. Just as we allowed the light of the colour to be measured to fall through three different filters on three photo-electric cells and then combined the three photo-electric currents produced in order to define the colour in colour space, so we might have taken three photographs with the aid of different coloured filters and then used these photographs in combination to produce colour sensations, forming as good a reproduction as possible as that of the original object.

The simplest method of reproduction, in principle, is that of additive combination. In the colour film this consists of making three differently coloured pictures of the object, with the aid of the three photographs (a red, green and blue picture), and uniting them again to one picture by additive colour mixture. The colour of each part of this picture is then determined by the contribution of the three separate pictures. For additive mixing all kinds of methods can be used, of which most have already been mentioned in section 59. Thus, with the help of prisms and mirrors, we can unite the rays proceeding from each of the separate pictures; a picture can also be produced that is built up of very tiny particles of red, green and

blue images; the eye itself then supplies the additive mixing. Finally we can show the three coloured pictures separately in swift succession (cinema, television). In this case, too, the mixing takes place in the eye of the observer. These additive methods are at first sight so attractive because the various steps of the process can be followed and calculated fairly easily (owing to the simple laws of additive mixing). By such calculations we can also draw up the conditions with which the filters, the photographic material and the lights used for the red, green and blue picture must comply in order to obtain a natural reproduction of all colours of the scene. "Natural" includes also the condition that colours which in the original radiate light of different spectral composition but make the same colour sensation must look the same in the picture. We see therefore that the above conditions for the filters etc. must be closely connected with the distribution curves of the eye. This connection is, in spite of technical differences, in principle the same for all additive processes, and in the subsequent mathematical calculations the same set-back is always met with: the demand for a completely faithful reproduction leads automatically to the absurd result that for certain spectral colours either the spectral transmission factor of a filter or the sensitivity of the photographic material must be negative [H a r d y [4])]. This impossible requirement forms a serious obstacle in the way of a perfect reproduction. We shall mention some attempts to remove or to evade this obstacle.

a. The occurrence of the negative values goes hand in hand with the impossibility of matching all colours in the colour triangle by additive mixture of a fixed triplet of colours without making use of a "negative quantity of a colour" (section 21). It is possible, however, to limit the necessity for negative transmission factors and sensitivities to a minimum by a favourable selection of the three coloured illuminants producing the red, green and blue image. (The triangle RGB, formed by the colour points of these lights must contain as large a part as possible of all colours of the colour plane.)

b. If the negative parts have been made as small as possible we can try to substitute the value 0 for all the negative values required by theory. This, of course, causes deviations from the naturalness of the reproduction: the colours do not quite correspond either in hue or saturation with the original. Deviations in hue are the most unpleasant.

c. It can be so arranged that while avoiding negative values the colour deviations are almost entirely limited to saturation [see for instance M a c A d a m [4])] and become therefore less noticeable.

d. There are various possibilities for meeting the necessity of negative values. For this purpose a subtraction of colours must take place in a certain sense. If there is in the reproduction process some intermediate stage in which we have electric potential differences to deal with instead of colours (television) the possibility will exist, in principle, of letting this subtraction take place in the shape of voltages with a reverse sign. In the colour film and in colour photography we can perform the subtraction to a certain degree by correcting the negative in printing by a weak positive placed before it [M a c A d a m [4])]. In some cases it is possible, with an eye to negative values, to make use of special properties of the photographic material (H e r s c h e l effect, curving of the density curve) but such methods are far less generally applicable than the other. On the whole, attempts to satisfy the demands for negative values only make the various processes more intricate.

§ 93 *Colour reproduction; subtractive method*

Besides the methods of colour reproduction mentioned there is a large number of processes in which subtractive mixture occurs. In colour printing this takes place by printing several coloured photographs on the paper on top of each other, so that in considering the final result each light ray must penetrate different layers of pigment in succession before the light — in a coloured state — reaches the eye. In the colour film the subtractive colour mixture takes place at a slightly earlier stage. While in the additive reproduction the image is built up by a mixing of three coloured pictures the coloured picture in these methods is formed by allowing the light — in this case white light — to pass in succession through three photographs placed one behind the other.
In some methods applied in colour printing it is often, at first sight, not clear whether the method employed is additive or subtractive. Take for example the process in which the separate coloured photographs of each part of the scene consists of a very fine network of coloured points *(fig. 111)*. These three coloured photographs are printed on top of one another, and the colour produced by this is

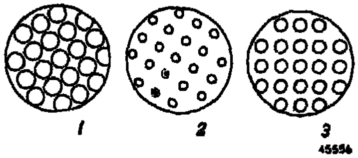


Fig. 111
The different gratings of dots are superimposed in printing. The ratios of the diameters determine the final colour.

determined by the ratio of the size of the points in the three coloured photographs.

Now during printing, the following events may occur: 1. a point of the paper will receive no colour; 2. it will receive only one colour; 3. it will receive two overlapping colours and 4. it will be coloured by three overlapping points. *Fig. 112* illustrates these four possibilities. The eight parts given in fig. 112 by numbers all show different colours. Some of these colours are produced by subtractive mixture (namely colours 3 and 4). The whole picture therefore contains only eight colours but on looking at it, an additive mixture of those eight colours takes place in the eye (because the network of points is so extremely fine), whereby the resulting colour is determined by the relative size of the three kinds of coloured points. [For a theoretical treatment of this case see N e u g e b a u e r [1])].

With purely subtractive methods of reproduction (as is often the case in colour films) it is very difficult to get a complete theoretical view of the course of the different processes. The difficulty is chiefly due to the fact that the process of subtractive colour mixture is governed by much less simple laws than that of additive mixing. In attempts to set up an exact theory [Y u l e [1, 3])] or to find a practical method of approximate calculation [M a c A d a m [5])] it appears that the properties of the photographic material must be taken into account to a much greater extent than was the case with additive methods. The same difficulties regarding negative values as discussed in section 92 appear to arise here too; the best method to overcome them is the method given under (d) of correction of the negatives by a weak positive. The only case in which the calculation of the subtractive reproduction becomes as simple as with the additive methods, occurs when each spectral colour is partially absorbed by only one of the pigments used for the photographs placed the one over the other while the other two transmit the light unimpaired. This will occur if the transmission curves of the three pigments have shapes as drawn in *fig. 113* [see H a r d y [4])].

Only in this special case is the influence of a modification of one of the filters independent of the other filters.

Subtractive colour mixture complies approximately with simple rules when it is a question of passing successively through filters which only absorb very little. Let W be the point in the colour space of the unfiltered

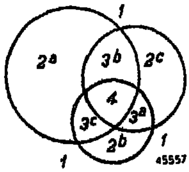


Fig. 112
The 8 "primary colours", arising from the method of fig. 111, which are additively mixed by the eye.

light, W_1 and W_2 the points of the light when passing only through the first (or the second) filter, point W_{12} (the light transmitted by both filters in succession) can then be found by vector addition: $WW_1W_2W_{12}$ is approximately a parallelogram see also [Neugebauer[1]].

Although it is theoretically much more difficult with subtractive methods to give the conditions for a correct reproduction than it is with additive methods, yet we find the subtractive principle far more frequently employed in practice. In order to understand this, we must remember that, in practice, a compromise must always be found between many often contradictory desires and demands. In making this compromise the following points must be considered (with special reference to the film):

a. The technical possibilities of obtaining a "faithful" reproduction.
b. The expense of the various processes.
c. The degree of intricacy of the reproducing apparatus.
d. The degree of divergence the colours show when using an imperfect system.
e. How unpleasant are such divergences?
f. What colours occur most in the scenes to be filmed?
g. Is a perfect copy of the reality required or desired?

We shall discuss one or two of these points.

The frequent use of the subtractive systems is chiefly due to points a-c. The point (d) has been dealt with by Yule[2]).

Point (e) brings us once more to the complicated problem of tolerances. Here it is of the greatest importance that the observer has some idea of the colours of the original. A colour divergence which is not conspicuous in a photograph of an interior may be very unpleasant in a photograph in the open air. Allowances must also be made for the fact that a part of the colour divergences can be partially compensated by chromatic adaptation.

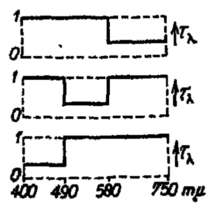


Fig. 113
This figure represents the spectral transmission of three pigment colours whose subtractive mixing gives simple results.

For point (f) the correct reproduction of the colours must be taken into account in making one's demands. Thus we have often seen that yellow colours — indistinguishable to the eye — may have been produced from a very narrow portion of the spectrum, or by combination of red, yellow

and yellowish green rays. As, in practice, the second manner of production occurs practically exclusively, we need not keep too strictly to the rule in this case, that two similar colours must be reproduced in a similar manner.

In considering point (g) we must remember that the picture is seen under quite different circumstances from the original. As regards the environment of the picture, with coloured pictures in a book there is usually a white background, the brightness of which is therefore greater than the average brightness of the picture, while with colour films, on the other hand, the environment is quite dark. Such differences do not influence the colour of a picture but they can be of great importance for the production of colour sensations. Thus most colours when observed in a relatively small field of vision with dark surroundings produce a more saturated impression than when seen under other circumstances. Hence we must draw the apparently rather strange conclusion that the most accurate reproduction of the original does not always produce the most natural impression! [Lagorio [1])].

Surveying the field of colour reproduction as a whole it strikes us how complicated it actually is and how many factors arising from different branches of science play a part. In this respect the last subject considered by us is a faithful picture of the general theory of colour which we have also learnt to know as the borderland between different sciences.

For further enlightenment in the matter of colour reproduction we must refer the reader to the special literature on this subject. Besides the references already quoted further information is to be found inthe works of Schrödinger [7]) (section 37), Klein [1]), Eggert [1]), MacAdam [7]), Evans [1]), Custers [1]) etc. May the short survey given here, as this book itself, serve to facilitate the further study of literature of the subject.

Appendix

I. TABLES FOR THE CALCULATION OF COLOURS

TABLE A Complementary wavelengths with respect to standard white E

λ	λ′	λ	λ′	λ	λ′	λ	λ′
380	569·7	480	580·4	570	427·4	590	486·8
400	569·7	481	581·4	571	447·6	591	487·2
410	569·8	482	582·5	572	457·1	592	487·6
420	569·8	483	583·7	573	463·5	593	488·0
430	570·1	484	585·0	574	467·8	594	488·3
440	570·5	485	586·6	575	470·8	595	488·7
450	571·2	486	588·4	576	473·3	596	489·0
455	571·7	487	590·6	577	475·2	597	489·2
460	572·4	488	593·0	578	476·9	598	489·5
465	573·3	489	596·1	579	478·3	599	489·7
470	574·7	490	600·0	580	479·6	600	490·0
471	575·1	491	605·1	581	480·6	610	491·7
472	575·5	492	612·8	582	481·6	620	492·6
473	575·9	493	627·3	583	482·5	630	493·1
474	576·4			584	483·3	640	493·5
475	576·9			585	484·0	650	493·6
476	577·4			586	484·6	660	493·8
477	578·1			587	485·2	680	493·8
478	578·8			588	485·8	700	493·9
479	579·5			589	486·3	780	493·9

TABLE B Tristimulus values for the equal energy spectrum and coordinates in the colour plane for $B_1B_2B_3$ system (Ch. IV)

$\bar{B}$	$\bar{B}_2$	$\bar{B}_3$	λ	b_1	b_2	b_3
0·0000	—0·0001	0·0001	**380**	+0·810	—1·573	1·763
0·0001	—0·0020	0·0002	**390**	0·792	—1·585	1·794
0·0003	—0·0006	0·0007	**400**	0·758	—1·577	1·819
0·0008	—0·0019	0·0022	**410**	0·705	—1·567	1·862
0·0021	—0·0050	0·0069	**420**	0·528	—1·264	1·736
0·0022	—0·0055	0·0149	**430**	0·188	—0·470	1·283
—0·0026	+0·0068	0·0188	**440**	—0·113	0·297	0·816
—0·0121	0·0311	0·0190	**450**	—0·319	0·818	0·501
—0·0261	0·0682	0·0179	**460**	—0·435	1·136	0·299
—0·0393	0·1165	0·0138	**470**	—0·432	1·280	0·152
—0·0494	0·1797	0·0087	**480**	—0·355	1·293	0·063
—0·0581	0·2612	0·0050	**490**	—0·280	1·256	0·024
—0·0717	0·3919	0·0029	**500**	—0·222	1·213	0·009
—0·0890	0·5904	0·0016	**510**	—0·177	1·174	0·003
—0·0926	0·8019	0·0007	**520**	—0·130	1·129	0·001
—0·0710	0·9327	0·0003	**530**	—0·082	1·082	0·000
—0·0315	0·9854	0·0001	**540**	—0·033	1·033	
+0·0228	0·9722	—0·0000	**550**	+0·023	0·977	
0·0906	0·9045	—0·0001	**560**	0·091	0·909	
0·1677	0·7844	—0·0001	**570**	0·176	0·824	
0·2453	0·6248	0·0000	**580**	0·282	0·718	
0·3093	0·4478		**590**	0·409	0·591	
0·3443	0·2867		**600**	0·546	0·454	
0·3397	0·1633		**610**	0·675	0·325	
0·2971	0·0839		**620**	0·780	0·220	
0·2268	0·0382		**630**	0·856	0·144	
0·1597	0·0153		**640**	0·912	0·088	
0·1017	0·0053		**650**	0·950	0·050	
0·0593	0·0017		**660**	0·972	0·028	
0·0315	0·0005		**670**	0·984	0·016	
0·0169	0·0001		**680**	0·992	0·008	
0·0082	0·0000		**690**	0·998	0·002	
0·0041			**700**	1·000	0·000	
0·0021			**710**	1·000		
0·0010			**720**	1·000		
0·0005			**730**	1·000		
0·0002			**740**	1·000		
0·0001			**750**	1·000		
0·0001			**760**	1·000		
0·0000			**770**	1·000		
Σ 1·891	8·681	0·1136				

TABLE C

RGB system (C.I.E. 1931) Tristimulus values of the equal energy spectrum and coordinates of spectral colours in the colour plane

$\bar{R}$	$\bar{G}$	$\bar{B}$	λ	r	g	b
0·0000	—0·0000	0·0012	**380**	0·0272	—0·0115	0·9843
0·0001	—0·0000	0·0036	**390**	0·0263	—0·0114	0·9851
0·0003	—0·0001	0·0121	**400**	0·0247	—0·0112	0·9865
0·0008	—0·0004	0·0371	**410**	0·0225	—0·0109	0·9884
0·0021	—0·0011	0·1154	**420**	0·0181	—0·0094	0·9913
0·0022	—0·0012	0·2477	**430**	0·0088	—0·0048	0·9960
—0·0026	0·0015	0·3123	**440**	—0·0084	0·0048	1·0036
—0·0121	0·0068	0·3167	**450**	—0·0390	0·0218	1·0172
—0·0261	0·0148	0·2982	**460**	—0·0909	0·0517	1·0392
—0·0393	0·0254	0·2299	**470**	—0·1821	0·1175	1·0646
—0·0494	0·0391	0·1449	**480**	—0·3667	0·2906	1·0761
—0·0581	0·0569	0·0826	**490**	—0·7150	0·6996	1·0154
—0·0717	0·0854	0·0478	**500**	—1·1685	1·3905	0·7780
—0·0890	0·1286	0·0270	**510**	—1·3371	1·9318	0·4053
—0·0926	0·1747	0·0122	**520**	—0·9830	1·8534	0·1296
—0·0710	0·2032	0·0055	**530**	—0·5159	1·4761	0·0398
—0·0315	0·2147	0·0015	**540**	—0·1707	1·1628	0·0079
0·0228	0·2118	—0·0006	**550**	0·0974	0·9051	—0·0025
0·0906	0·1970	—0·0013	**560**	0·3164	0·6881	—0·0045
0·1677	0·1709	—0·0014	**570**	0·4973	0·5067	—0·0040
0·2453	0·1361	—0·0011	**580**	0·6449	0·3579	—0·0028
0·3093	0·0975	—0·0008	**590**	0·7617	0·2402	—0·0019
0·3443	0·0625	—0·0005	**600**	0·8475	0·1537	—0·0012
0·3397	0·0356	—0·0003	**610**	0·9059	0·0949	—0·0008
0·2971	0·0183	—0·0002	**120**	0·9425	0·0580	—0·0005
0·2268	0·0083	—0·0001	**630**	0·9649	0·0354	—0·0003
0·1597	0·0033	—0·0000	**640**	0·9797	0·0205	—0·0002
0·1017	0·0012		**650**	0·9888	0·0113	—0·0001
0·0593	0·0004		**660**	0·9940	0·0061	—0·0001
0·0315	0·0001		**670**	0·9966	0·0035	—0·0001
0·0169	0·0000		**680**	0·9984	0·0016	—0·0000
0·0082			**690**	0·9996	0·0004	
0·0041			**700**	1·0000	0·0000	
0·0021			**710**	1·0000		
0·0010			**720**	1·0000		
0·0005			**730**	1·0000		
0·0002			**740**	1·0000		
0·0001			**750**	1·0000		
0·0001			**760**	1·0000		
0·0000			**770**	1·0000		
1·891	1·891	1·889				

TABLE D

XYZ system (C.I.E. 1931): Tristimulus values of the equal energy spectrum

λ	$\bar{X}$	$\bar{Y}=V_\lambda$	$\bar{Z}$	λ	$\bar{X}$	$\bar{Y}=V_\lambda$	$\bar{Z}$	λ	$\bar{X}$	$\bar{Y}=V_\lambda$	$\bar{Z}$
				525	0·1096	0·7932	0·0573	675	0·0636	0·0232	
380	0·0014	0·0000	0·0065	530	0·1655	0·8620	0·0422	680	0·0468	0·0170	
385	0·0022	0·0001	0·0105	535	0·2257	0·9149	0·0298	685	0·0329	0·0119	
390	0·0042	0·0001	0·0201	540	0·2904	0·9540	0·0206	690	0·0227	0·0082	
395	0·0076	0·0002	0·0362	545	0·3597	0·9803	0·0134	695	0·0158	0·0057	
400	0·0143	0·0004	0·0679	550	0·4334	0·9950	0·0087	700	0·0114	0·0041	
405	0·0232	0·0006	0·1102	555	0·5121	1·0002	0·0057	705	0·0081	0·0029	
410	0·0435	0·0012	0·2074	560	0·5945	0·9950	0·0039	710	0·0058	0·0021	
415	0·0776	0·0022	0·3713	565	0·6784	0·9786	0·0027	715	0·0041	0·0015	
420	0·1344	0·0040	0·6459	570	0·7621	0·9520	0·0021	720	0·0029	0·0010	
425	0·2148	0·0073	1·0391	575	0·8425	0·9154	0·0018	725	0·0020	0·0007	
430	0·2839	0·0116	1·3856	580	0·9163	0·8700	0·0017	730	0·0014	0·0005	
435	0·3285	0·0168	1·6230	585	0·9786	0·8163	0·0014	735	0·0010	0·0004	
440	0·3483	0·0230	1·7471	590	1·0263	0·7570	0·0011	740	0·0007	0·0003	
445	0·3481	0·0298	1·7826	595	1·0567	0·6949	0·0010	745	0·0005	0·0002	
450	0·3362	0·0380	1·7721	600	1·0622	0·6310	0·0008	750	0·0003	0·0001	
455	0·3187	0·0480	1·7441	605	1·0456	0·5668	0·0006	755	0·0002	0·0001	
460	0·2908	0·0600	1·6692	610	1·0026	0·5030	0·0003	760	0·0002	0·0001	
465	0·2511	0·0739	1·5281	615	0·9384	0·4412	0·0002	765	0·0001	0·0000	
470	0·1954	0·0910	1·2876	620	0·8544	0·3810	0·0002	770	0·0001	0·0000	
475	0·1421	0·1126	1·0419	625	0·7514	0·3210	0·0001	775	0·0000	0·0000	0·0000
480	0·0956	0·1390	0·8130	630	0·6424	0·2650	0·0000				
485	0·0580	0·1693	0·6162	635	0·5419	0·2170	0·0000				
490	0·0320	0·2080	0·4652	640	0·4479	0·1750	0·0000				
495	0·0147	0·2586	0·3533	645	0·3608	0·1382	0·0000				
500	0·0049	0·3230	0·2720	650	0·2835	0·1070	0·0000				
505	0·0024	0·4073	0·2123	655	0·2187	0·0816					
510	0·0093	0·5030	0·1582	660	0·1649	0·0610					
515	0·0291	0·6082	0·1117	665	0·1212	0·0446					
520	0·0633	0·7100	0·0782	670	0·0874	0·0320					
								Σ	21·37	21·37	21·37

TABLE E

XYZ system (C.I.E.1931) Trichromatic coefficients (coordinates in the colour plane) of the spectral colours

λ	*x*	*y*	*z*	λ	*x*	*y*	*z*	λ	*x*	*y*
				525	0·1142	0·8262	0·0596	675	0·7327	0·2673
380	0·1741	0·0050	0·8209	530	0·1547	0·8059	0·0394	680	0·7334	0·2666
385	0·1740	0·0050	0·8211	535	0·1920	0·7816	0·0255	685	0·7340	0·2660
390	0·1738	0·0049	0·8213	540	0·2296	0·7543	0·0161	690	0·7344	0·2656
395	0·1736	0·0049	0·8215	545	0·2658	0·7243	0·0099	695	0·7346	0·2654
400	0·1733	0·0048	0·8219	550	0·3016	0·6923	0·0061	700	0·7347	0·2653
405	0·1730	0·0048	0·8222	555	0·3373	0·6589	0·0038	705		0·2653
410	0·1726	0·0048	0·8226	560	0·3731	0·6245	0·0024	710		0·2653
415	0·1721	0·0048	0·8231	565	0·4087	0·5896	0·0017	715		0·2653
420	0·1714	0·0051	0·8235	570	0·4441	0·5547	0·0012	720		0·2653
425	0·1703	0·0058	0·8239	575	0·4788	0·5202	0·0010	725		0·2653
430	0·1689	0·0069	0·8242	580	0·5125	0·4866	0·0009	730		0·2653
435	0·1669	0·0086	0·8245	585	0·5448	0·4544	0·0008	735		0·2653
440	0·1644	0·0109	0·8247	590	0·5752	0·4242	0·0006	740		0·2653
445	0·1611	0·0138	0·8251	595	0·6029	0·3965	0·0006	745		0·2653
450	0·1566	0·0177	0·8257	600	0·6270	0·3725	0·0005	750		0·2653
455	0·1510	0·0227	0·8263	605	0·6482	0·3514	0·0004	755		0·2653
460	0·1440	0·0297	0·8263	610	0·6658	0·3340	0·0002	760		0·2653
465	0·1355	0·0399	0·8246	615	0·6801	0·3197	0·0002	765		0·2653
470	0·1241	0·0578	0·8181	620	0·6915	0·3083	0·0002	770		0·2653
475	0·1096	0·0868	0·8036	625	0·7006	0·2993	0·0001	775		0·2653
480	0·0913	0·1327	0·7760	630	0·7079	0·2920	0·0001	780	0·7347	0·2653
485	0·0687	0·2007	0·7306	635	0·7140	0·2859	0·0001			
490	0·0454	0·2950	0·6596	640	0·7190	0·2810	0·0001			
495	0·0235	0·4127	0·5638	645	0·7230	0·2770	0·0000			
500	0·0082	0·5384	0·4534	650	0·7260	0·2740				
505	0·0039	0·6548	0·3413	655	0·7283	0·2717				
510	0·0139	0·7502	0·2359	660	0·7300	0·2700				
515	0·0389	0·8120	0·1491	665	0·7311	0·2689				
520	0·0743	0·8338	0·0919	670	0·7320	0·2680				

TABLE F

XYZ system (C.I.E. 1931): Spectral energy distribution of standard illuminants A, B, and C

λ	E_A	E_B	E_C	λ	E_A	E_B	E_C	λ	E_A	E_B	E_C
				525	75·79	90·43	96·78	675	182·1	104·6	85·30
380	9·79	22·40	33·00	530	79·13	92·20	98·00	680	185·4	103·9	84·00
385	10·90	26·85	39·92	535	82·52	94·46	99·94	685	188·7	102·8	82·21
390	12·09	31·30	47·40	540	85·95	96·90	102·1	690	191·9	101·6	80·20
395	13·36	36·18	55·17	545	89·41	99·16	104·0	695	195·1	100·4	78·24
400	14·71	41·30	63·30	550	92·91	101·0	105·2	700	198·3	99·10	76·30
405	16·15	46·62	71·81	555	96·44	102·2	105·7	705	201·4	97·70	74·36
410	17·68	52·10	80·60	560	100·0	102·8	105·3	710	204·4	96·20	72·40
415	19·29	57·70	89·53	565	103·6	102·9	104·1	715	207·4	94·60	70·40
420	21·00	63·20	98·10	570	107·2	102·6	102·3	720	210·4	92·90	68·30
425	22·79	68·37	105·8	575	110·8	101·9	100·2	725	213·3	91·10	66·30
430	24·67	73·10	112·4	580	114·4	101·0	97·80	730	216·1	89·40	64·40
435	26·64	77·31	117·8	585	118·1	100·1	95·43	735	218·9	88·00	62·80
440	28·70	80·80	121·5	590	121·7	99·20	93·20	740	221·7	86·90	61·50
445	30·85	83·44	123·5	595	125·4	98·44	91·22	745	224·4	85·90	60·20
450	33·09	85·40	124·0	600	129·0	98·00	89·70	750	227·0	85·20	59·20
455	35·41	86·88	123·6	605	132·7	98·08	88·83	755	229·6	84·80	58·50
460	37·82	88·30	123·1	610	136·3	98·50	88·40	760	232·1	84·70	58·10
465	40·30	90·08	123·3	615	140·0	99·06	88·19	765	234·6	84·90	58·00
470	42·87	92·00	123·8	620	143·6	99·70	88·10	770	237·0	85·40	58·20
475	45·52	93·75	124·1	625	147·2	100·4	88·06	775	239·4	86·10	58·50
480	48·25	95·20	123·9	630	150·8	101·0	88·00	780	241·7	87·00	59·10
485	51·04	96·23	122·9	635	154·4	101·6	87·86				
490	53·91	96·50	120·7	640	158·0	102·2	87·80				
495	56·85	95·71	116·9	645	161·5	103·0	87·99				
500	59·86	94·20	112·1	650	165·0	103·9	88·20				
505	62·93	92·37	107·0	655	168·5	104·6	88·20				
510	66·06	90·70	102·3	660	172·0	105·0	87·90				
515	69·25	89·65	98·81	665	175·4	105·1	87·22				
520	72·50	89·50	96·90	670	178·8	104·9	86·30				

XYZ system (C.I.E. 1931): Tristimulus values of spectral colours

	TABLE G Illuminant A			TABLE H Illuminant B			TABLE I Illuminant C			
λ	$E_A\bar{X}$	$E_A\bar{Y}$	$E_A\bar{Z}$	$E_B\bar{X}$	$E_B\bar{Y}$	$E_B\bar{Z}$	$E_C\bar{X}$	$E_C\bar{Y}$	$E_C\bar{Z}$	λ
380	0·01	0·00	0·06	0·03	0·00	0·15	0·05	0·00	0·22	**380**
390	0·05	0·00	0·24	0·13	0·00	0·63	0·20	0·01	0·95	**390**
400	0·21	0·01	1·01	0·60	0·02	2·83	0·91	0·02	4·34	**400**
410	0·76	0·02	3·64	2·25	0·06	10·71	3·48	0·10	16·58	**410**
420	2·82	0·08	13·56	8·50	0·25	40·82	13·19	0·39	63·36	**420**
430	7·00	0·29	34·18	20·76	0·85	101·29	31·02	1·30	155·74	**430**
440	9·96	0·66	49·94	28·03	1·86	140·61	42·15	2·79	211·43	**440**
450	11·12	1·26	58·66	28·71	3·24	151·39	41·69	4·71	219·81	**450**
460	11·00	2·27	63·13	25·69	5·30	147·40	35·81	7·39	205·49	**460**
470	8·38	3·90	55·22	17·98	8·37	118·50	24·19	11·27	159·46	**470**
480	4·61	6·71	39·22	9·10	13·23	77·38	11·85	17·22	100·71	**480**
490	1·73	11·21	25·07	3·09	20·07	44·88	3·86	25·11	56·13	**490**
500	0·29	19·34	16·28	0·46	30·43	26·62	0·55	36·21	30·49	**500**
510	0·62	33·23	10·45	0·85	45·62	14·35	0·95	51·46	16·18	**510**
520	4·59	51·48	5·67	5·66	63·54	7·00	6·13	68·80	7·58	**520**
530	13·09	68·21	3·34	15·26	79·48	3·89	16·22	84·48	4·13	**530**
540	24·96	82·00	1·75	28·14	92·44	1·97	29·65	97·40	2·08	**140**
550	40·27	92·44	0·82	43·78	100·50	0·88	45·60	104·67	0·92	**550**
560	59·45	99·50	0·39	61·11	102·29	0·40	62·06	104·77	0·41	**560**
570	81·69	102·04	0·22	78·20	97·68	0·21	77·97	97·39	0·21	**570**
580	104·85	99·56	0·18	92·54	87·87	0·16	89·61	85·09	0·15	**580**
590	124·97	92·15	0·14	101·84	75·09	0·12	95·68	70·55	0·11	**590**
600	137·04	81·42	0·10	104·07	61·84	0·08	95·26	56·60	0·07	**600**
610	136·73	68·58	0·05	98·78	49·55	0·04	88·65	44·46	0·03	**610**
620	122·72	54·72	0·03	85·19	37·99	0·02	75·28	33·57	0·02	**620**
630	96·90	39·97	0·01	64·89	26·76	0·01	56·54	23·32	0·01	**630**
640	70·75	27·65	0·01	45·77	17·88	0·00	39·32	15·36	0·00	**640**
650	46·79	17·66	0·00	29·46	11·12		25·01	9·44		**650**
660	28·36	10·49		17·32	6·40		14·50	5·36		**660**
670	15·62	5·72		9·17	3·36		7·54	2·76		**670**
680	8·67	3·15		4·86	1·77		3·93	1·43		**680**
690	4·35	1·57		2·30	0·83		1·82	0·66		**690**
700	2·25	0·81		1·12	0·41		0·87	0·31		**700**
710	1·19	0·43		0·56	0·20		0·42	0·15		**710**
720	0·61	0·22		0·27	0·10		0·20	0·07		**720**
730	0·31	0·11		0·13	0·05		0·09	0·03		**730**
740	0·15	0·06		0·06	0·02		0·04	0·02		**740**
650	0·08	0·03		0·03	0·01		0·02	0·01		**750**
760	0·04	0·01		0·01	0·01		0·01	0·00		**760**
770	0·02	0·01		0·01	0·00		0·00	0·00		**770**
780	0·01	0·00		0·00	0·00		0·00	0·00		**780**
Σ	1185	1079	383·4	1038	1046^{5}	891·3	1044	1065	1257	
xyz	0·447	0·407^{5}	0·145	0·348^{5}	0·352	0·300	0·310	0·316	0·373^{5}	

TABLE J   *XYZ* system (C.I.E. 1931): Slopes of dominant wave-

λ	$\frac{x-x_w}{y-y_w}$	λ	$\frac{x-x_w}{y-y_w}$	λ	$\frac{x-x_w}{y-y_w}$	λ	$\frac{x-x_w}{y-y_w}$	$\frac{y-y_w}{x-x_w}$	λ	$\frac{x-x_w}{y-y_w}$	$\frac{y-y_w}{x-x_w}$
380	0·48508	410	0·4893	440	0·5240	470	0·7594		500		—0·6304
381	0·48513	411	0·4897	441	0·5267	471	0·7825		501		0·7013
382	0·48517	412	0·4900	442	0·5296	472	0·8084		502		0·7714
383	0·48525	413	0·4903	443	0·5323	473	0·8372		503		0·8403
384	0·48532	414	0·4906	444	0·5354	474	0·8699		504		0·9081
385	0·48537	415	0·4909	445	0·5391	475	0·9075		505	—1·0252	—0·9754
386	0·48548	416	0·4913	446	0·5428	476	0·9510	1·0515	506	0·9594	—1·0423
387	0·48555	417	0·4916	447	0·5465	477	1·0009	0·9991	507	0·9021	
388	0·48563	418	0·4924	448	0·5507	478		0·9449	508	0·8516	
389	0·48574	419	0·4927	449	0·5555	479		0·8883	509	0·8068	
390	0·48584	420	0·4935	450	0·5600	480		0·8290	510	—0·7666	
391	0·48595	421	0·4942	451	0·5648	481		0·7670	511	0·7304	
392	0·48606	422	0·4950	452	0·5701	482		0·7033	512	0·6977	
393	0·48620	423	0·4959	453	0·5753	483		0·6374	513	0·6677	
394	0·48633	424	0·4966	454	0·5811	484		0·5704	514	0·6403	
395	0·48643	425	0·4979	455	0·5871	485		0·5013	515	—0·6153	
396	0·48659	426	0·4991	456	0·5935	486		0·4302	516	0·5928	
397	0·48673	427	0·5000	457	0·6003	487		0·3577	517	0·5722	
398	0·48687	428	0·5012	458	0·6079	488		0·2838	518	0·5528	
399	0·48705	429	0·5024	459	0·6155	489		0·2089	519	0·5347	
400	0·48722	430	0·5038	460	0·6236	490		0·1333	520	—0·5178	
401	0·48740	431	0·5055	461	0·6319	491		0·0560	521	0·5019	
402	0·48757	432	0·5072	462	0·6410	492		—0·0225	522	0·4870	
403	0·48774	433	0·5086	463	0·6510	493		—0·1008	523	0·4726	
404	0·48792	434	0·5108	464	0·6622	494		—0·1785	524	0·4587	
405	0·48813	435	0·5126	465	0·6743	495		—0·2559	525	—0·4448	
406	0·48830	436	0·5148	466	0·6877	496		0·3329	526	0·4313	
407	0·48854	437	0·5167	467	0·7030	497		0·4087	527	0·4177	
408	0·48877	438	0·5190	468	0·7200	498		0·4838	528	0·4045	
409	0·48906	439	0·5212	469	0·7382	499		0·5574	529	0·3913	

length lines with respect to standard white E (sect. 40)

λ	$\frac{x-x_w}{y-y_w}$	λ	$\frac{x-x_w}{y-y_w}$	$\frac{y-y_w}{x-x_w}$	λ	$\frac{y-y_w}{x-x_w}$	λ	$\frac{y-y_w}{x-x_w}$	λ	$\frac{y-y_w}{x-x_w}$	λ	$\frac{y-y_w}{x-x_w}$
530	—0·3782	**560**	0·1364		**590**	0·3755	**620**	—0·0701	650	—0·1513	**680**	—0·16700
531	0·3652	**561**	0·1647		**591**	0·3433	**621**	0·0750	651	0·1524	**681**	0·16725
532	0·3523	**562**	0·1949		**592**	0·3136	**622**	0·0798	652	0·1535	**682**	0·16750
533	0·3394	**563**	0·2261		**593**	0·2852	**623**	0·0844	653	0·1543	**683**	0·16773
534	0·3264	**564**	0·2591		**594**	0·2589	**624**	0·0886	654	0·1554	**684**	0·16796
535	—0·3135	**565**	0·2939		**595**	0·2341	**625**	—0·0929	655	—0·1562	**685**	—0·16819
536	0·3005	**566**	0·3307		**596**	0·2112	**626**	0·0968	656	0·1571	**686**	0·16839
537	0·2872	**567**	0·3695		**597**	0·1899	**627**	0·1005	657	0·1577	**687**	0·16858
538	0·2737	**568**	0·4107		**598**	0·1698	**628**	0·1038	658	0·1586	**688**	0·16877
539	0·2602	**569**	0·4544		**599**	0·1508	**629**	0·1074	659	0·1592	**689**	0·16893
540	—0·2466	**570**	0·5002		**600**	0·1332	**630**	—0·1105	660	—0·1599	**690**	—0·16908
541	0·2325	**571**	0·5485		**601**	0·1161	**631**	0·1136	661	0·1603	**691**	0·16920
542	0·2182	**572**	0·6005		**602**	0·1002	**632**	0·1167	662	0·1609	**692**	0·16931
543	0·2034	**573**	0·6564		**603**	0·0847	**633**	0·1195	663	0·1613	**693**	0·16939
544	0·1884	**574**	0·7154		**604**	0·0704	**634**	0·1223	664	0·1618	**694**	0·16947
545	—0·1729	**575**	0·7784		**605**	0·0572	**635**	—0·1248	665	—0·1622	**695**	—0·16951
546	0·1573	**576**	0·8456		**606**	0·0449	**636**	0·1273	666	0·1626	**696**	0·16957
547	0·1409	**577**	0·9180		**607**	0·0329	**637**	0·1296	667	0·1630	**697**	0·16960
548	0·1239	**578**	0·9952	1·0048	**608**	0·0218	**638**	0·1319	668	0·1634	**698**	0·16962
549	0·1067	**579**	1·0788	0·9269	**609**	0·0115	**639**	0·1341	669	0·1637	**699**	0·16964
550	—0·0886	**580**		0·8554	**610**	0·0018	**640**	—0·1362	670	—0·1641		
551	—0·0698	**581**		0·7894	**611**	—0·0075	**641**	0·1382	671	0·1643		
552	—0·0506	**582**		0·7289	**612**	—0·0162	**642**	0·1399	672	0·1647		
553	—0·0304	**583**		0·6729	**613**	—0·0243	**643**	0·1417	673	0·1649		
554	—0·0096	**584**		0·6207	**614**	—0·0320	**644**	0·1432	674	0·1653		
555	+0·0120	**585**		0·5724	**615**	—0·0395	**645**	—0·1448	675	—0·1655		
556	0·0348	**586**		0·5276	**616**	0·0464	**646**	0·1463	676	0·1660		
557	0·0587	**587**		0·4857	**617**	0·0526	**647**	0·1476	677	0·1662		
558	0·0833	**588**		0·4463	**618**	0·0588	**648**	0·1489	678	0·1664		
559	0·1094	**589**		0·4101	**619**	0·0646	**649**	0·1502	679	0·1668		

TABLE K

XYZ system (C.I.E. 1931): Trichromatic coefficients of a total radiator (black body) at different temperatures

T	x	y	z	λ_d	T	x	y	z	λ_d
100	0·735	0·265	0·000	695	3000	0·436	0·404	0·160	582·9
300	0·734	0·266	0·000	684	3100	0·429	0·401	0·170	582·4
500	0·721	0·279	0·000	641·5	3200	0·423	0·399	0·178	582·1
600	0·708	0·291	0·001	630·8	3300	0·417	0·396	0·187	581·6
800	0·681	0·318	0·001	615·5	3400	0·411	0·393	0·196	581·3
1000	0·652	0·345	0·003	606·7	3500	0·405	0·390	0·205	580·9
1300	0·610	0·379	0·011	598·2	4000	0·380	0·377	0·243	578·9
1500	0·586	0·393	0·021	594·8	4500	0·360	0·364	0·276	576·8
1600	0·573	0·400	0·027	593·1	4800	0·351	0·356	0·293	574·8
1800	0·549	0·408	0·043	590·7	5000	0·345	0·351	0·304	573·2
2000	0·526	0·413	0·061	588·7	5500	0·332	0·341	0·327	
2100	0·515	0·415	0·070	587·8	6000	0·322	0·331	0·347	
2200	0·505	0·415	0·080	587·1	6500	0·313	0·323	0·363	485·2
2300	0·495	0·415	0·090	586·4	7000	0·306	0·316	0·494	483·7
2400	0·486	0·414	0·100	585·9	8000	0·295	0·305	0·400	481·0
2500	0·476	0·414	0·110	585·3	10000	0·280	0·288	0·431	479·1
2600	0·468	0·412	0·120	584·8	24000	0·253	0·253	0·493	477·0
2700	0·459	0·411	0·130	584·3	∞	0·240	0·234	0·526	475·8
2800	0·451	0·408	0·140	583·8					
2900	0·444	0·406	0·150	583·3	2848	0·4475	0·407^5	0·145	583·6

XYZ system (C.I.E. 1931): Selected wavelengths (sect. 51)

	TABLE L Standard white E ($x = 0.333$, $y = 0.333$)			TABLE M Illuminant A ($x = 0.4475$, $y = 0.4075$)			TABLE N Illuminant B ($x = 0.348$, $y = 0.352$)			TABLE O Illuminant C ($x = 0.310$, $y = 0.316$)			
	λ_x	λ_y	λ_z	λ_x	λ_y	λ_z	λ_x	λ_y	λ_z	λ_x	λ_y	λ_z	
1	423·7	469·7	410·4	444·0	487·8	416·4	428·1	472·3	414·8	424·4	465·9	414·1	1
2*	436·5	493·1	419·6	516·9	507·7	424·9	442·1	494·5	422·9	435·5	489·4	422·2	2*
3	446·8	504·6	424·1	544·0	517·3	429·4	454·1	505·7	427·1	443·9	500·4	426·3	3
4	457·7	512·1	427·4	554·2	524·1	432·9	468·1	513·5	430·3	452·1	508·7	429·4	4
5*	472·1	518·0	430·1	561·4	529·8	436·0	527·8	519·6	433·0	461·2	515·1	432·0	5*
6	531·6	522·9	432·5	567·1	534·8	438·7	543·3	524·8	435·4	473·9	520·6	434·3	6
7	544·9	527·4	434·8	572·0	539·4	441·3	551·9	529·4	437·7	531·0	525·4	436·5	7
8*	553·4	531·6	437·0	576·3	543·7	443·7	558·5	533·7	439·9	544·2	529·8	438·6	8*
9	560·0	535·5	438·5	580·2	547·8	446·0	564·0	537·7	442·0	552·3	533·9	440·6	9
10	565 5	539·3	441·1	583·9	551·7	448·3	568·8	541·5	444·0	558·7	537·7	442·5	10
11*	570·4	543·1	443·1	587·2	555·4	450·5	573·1	545·1	446·0	564·0	541·4	444·4	11*
12	574·8	546·7	445·1	590·5	559·1	452·6	577·1	548·7	448·0	568·8	544·9	446·3	12
13	578·9	550·3	447·1	593·5	562·7	454·7	580·9	552·1	450·0	573·2	548·3	448·2	13
14*	582·8	553·9	449·1	596·5	566·3	456·8	584·5	555 6	451·9	577·3	551·7	450·2	14*
15	586·4	557·4	451·1	599·4	569·8	458·8	588·0	559 0	453·9	581·2	555·1	452·1	15
16	589·9	561·0	453·2	602·3	573·3	460·8	591·4	562·4	455·8	585·0	558·5	454·0	16
17*	593·4	564·6	455·2	605·2	576·9	462·9	594·7	565·8	457·8	588·7	561·9	455·9	17*
18	596·7	568·3	457·2	608·0	580·5	464·9	598·1	569·3	459·8	592·3	565·3	457·9	18
19	600·1	572·0	459·3	610·9	584·1	467·0	601·4	572·9	461·8	595·9	568·8	459·9	19
20*	603·5	575·9	461·5	613·8	587·9	469·2	604·7	576·7	463·9	599·5	572·5	462·0	20*
21	606·9	579·9	463·7	617·0	591·8	471·6	608·1	580·6	466·1	603·2	576·4	464·1	21
22	610·4	584·1	466·0	620·0	595·9	474·1	611·6	584·7	468·4	606·9	580·4	466·4	22
23*	614·0	588·5	468·5	623·3	600·1	476·8	615·3	589·1	470·8	610·8	584·8	468·8	23*
24	617·9	593·3	471·3	626·9	604·7	479·9	619·1	593·9	473·6	614·9	589·5	471·4	24
25	622·1	598·5	474·3	630·8	609·7	483·4	623·3	599·1	476·6	619·2	594·8	474·4	25
26*	626·7	604·3	477·9	635·3	615 2	487·5	628·0	605·0	480·2	624·0	600·7	477·8	26*
27	632·2	611·0	482·4	640·5	621·5	492·7	633·4	611·8	484·5	629·6	607·6	481·9	27
28	638·8	619·1	488·2	646·9	629·2	499·3	640·1	619·9	490·2	636·4	616·0	487·3	28
29*	648·0	629·9	496·9	655·9	639·7	508·4	649·2	630·9	498·6	646·2	627·1	495·3	29*
30	665·4	649·7	513·7	673·5	659·0	526·7	666·3	650·7	515·2	662·2	647·0	511·5	30

II. LIST OF SYMBOLS

The numbers between brackets refer to sections

A,B,C	Illuminants (23)
Å	Ångström = $^1/_{10}$ mμ = 10^{-7} mm = 10^{-10} m (2)
a	Absorbing power (42)
a_{ij}	Coefficients of general linear transformation
B	Brightness (16, 17)
$B_1B_2B_3$	Names of primaries in the $B_1B_2B_3$ system; trichromatic coordinates in $B_1B_2B_3$ colour space
$\overline{B}_1,\overline{B}_2,\overline{B}_3$	Distribution curves of the equal-energy spectrum in the $B_1B_2B_3$ system (table B)
b_1,b_2,b_3	Trichromatic coefficients (coordinates in the colour plane) in the $B_1B_2B_3$ system (table B)
c_1,c_2	Radiation constants (P l a n c k's formula, 42)
E	Standard white; colour point of equal energy spectrum (23)
E	Power; energy radiated per second (12)
$E_\lambda \Delta\lambda$	Energy radiated in a continuous spectrum in the wavelength interval $\lambda \rightarrow \lambda + \Delta\lambda$.
e	2.71828.... (an important constant in higher mathematics)
I_1,I_2,I_3	The isochrome directions of the dichromats; directions of coordinate axes in R′ G' B' system (64.65)
k,w,z	Colour part, white part, black part in O s t w a l d's system (75)
L_1,L_2,L_3	Luminosity coefficients (32)
l	Distance between two division lines (light - dark) used in the description of boundary colours (45)
mμ	Milli-micron = 10^{-6} mm = 10^{-9} m (2)
N	Number indicating hue in O s t w a l d's system (75)
O	Origin of a coordinate system
p	Colorimetric purity (22, 35)
(RGB)	Trichromatic C.I.E. system (36)
R,G,B	Names of primaries in RGB system
R,G,B	Trichromatic coordinates in RGB colour space
$\overline{R},\overline{G},\overline{B}$	Distribution curves of the equal energy spectrum in RGB system (table C)
r,g,b	Trichromatic coefficients (coordinates in colour plane) in RGB system (table C)
$(R'G'B')$	Trichromatic system of sect. 65
R',G',B'	Trichromatic coordinates in $R'G'B'$ colour space
r',g',b'	Trichromatic coefficients (coordinates in colour plane) in $R'G'B'$ system
$(R''G''B'')$	J u d d's U.C.S. trichromatic system (81)
R'',G'',B''	Coordinates in J u d d's U.C.S. system
r'',g'',b''	Trichromatic coefficients (coordinates in colour plane) in U.S.C. system $(R''G''B'')$
r_λ	Spectral radiative power
T	Time of vibration (2), temperature (42)
(UVW)	M a c A d a m's trichromatic system
U,V,W	Trichromatic coordinates in UVW system
u,v,w	Trichromatic coefficients (coordinates in colour plane) in UVW system
V_λ	International relative luminosity (visibility) (16, table D)
$V_s V_k$	Relative luminosity for rod and cone vision
(XYZ)	Trichromatic system C.I.E. (ch. V)

X,Y,Z	Coordinate axes in XYZ system
XYZ	Trichromatic coordinates in the XYZ colour space
$\bar{X}\bar{Y}\bar{Z}$	Distribution curves of equal energy spectrum in XYZ system (table D)
$x,y,z,$	Trichromatic coefficients (coordinates in colour plane or chromaticity diagram) in XYZ system (table E)
x,y	Denotations of human sex chromosomes (67)
$(x'y'z')$	Trichromatic system (Breckenridge 81)
$x'y'z'$	Trichromatic coordinates in the $x'y'z'$ system
x,y,z	Trichromatic coefficients (coordinates in colour plane) in the $x'y'z'$ system
$x''y''z''$	Displaced coordinates in the $x'y'z'$ colour plane; $x'' = 0.075 - x'$; $y'' = y' - 0.5$

Δ	delta	Small increment of a quantity
ε	epsilon	Reflection factor (38)
ε_λ		Spectral reflection factor (38)
λ	lambda	Wavelength (in mμ) (2)
λ,λ'		Complementary wavelengths
λ_d		Dominant wavelength
μ	mu	Micron = $\frac{1}{1000}$ mm = 10^{-6} m (2)
ν	nu	Frequency (2)
π	pi	3.14159....
Σ	sigma	Sum of a number of terms
σ	sigma	Excitation purity (35)
τ	tau	Transmission factor
τ_λ		Spectral transmission factor (47)

c	Candle (unit of light)
c/m^2	Candle per square metre (unit of brightness) = $\frac{1}{10.000}$ stilb
lx	Lux (unit of illumination) = 1 lumen/m^2
lm	Lumen (unit of luminous flux)
sb	Stilb (unit of brightness) = 1 c/cm^2
C.I.E.	Commission Internationale d'Eclairage
I.B.K.	Internationale Beleuchtungs Kommission
I.C.I.	International Commission of Illumination
O.S.A.	Optical Society of America (16)
U.C.S.	Uniform chromaticity scale (5) (81)

$\longleftrightarrow$	Looks like, matches
♀	Female
♂	Male

III. LITERATURE

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